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Fuzzy ideals of BCI-algebras with respect to t-norm

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Abstract. In this paper, we introduce and define the concepts of fuzzy implicative ideals, fuzzy closed implicative ideals and fuzzy commutative ideals of BCI-algebras with respect to t-norms and we investigate some good examples. Next, we present some fundamental properties of them. Also, we link them with implicative ideals, closed implicative ideals and commutative ideals of BCI-algebras such that every fuzzy implicative ideal, fuzzy closed implicative ideals and fuzzy commutative ideals of BCI-algebras with respect to t-norms will be implicative ideals, closed implicative ideals and commutative ideals of BCI-algebras, respectively and we indicate some examples. Later we investigate and generalize the concepts as intersection and cartesian product of them and we show that the intersection and cartesian product of fuzzy implicative ideals, fuzzy closed implicative ideals and fuzzy commutative ideals of BCI-algebras with respect to t-norms are also fuzzy implicative ideals, fuzzy closed implicative ideals and fuzzy commutative ideals of BCI-algebras with respect to t-norms, respectively and characterize their properties of them with examples. Finally, we prove some properties about them under homomorphisms (image and pre-image) of BCI-algebras.

1. Introduction

BCI-algebras are established from two distinct approaches as propositional calculi and set theory. Iseki [4] introduced the idea of BCI-algebras. Fuzzy algebraic structures play a prominent role in different domains in mathematics and other sciences. Fuzzy set theory, initially established by Zadeh [14] in 1965, was applied by

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researchers to generalize some of the essential ideas of algebraic structures. Several authors investigated some properties of fuzzy BCI-algebras structures [**5, 11, 15**].

t-norms are operations which generalize the logical conjunction to fuzzy logic. In the previous work, we investigated some properties of fuzzy algebraic structures [**8, 9, 10**]. In this work, by using t-norms, fuzzy implicative ideals as $FIIT(X)$, fuzzy closed implicative ideals as $FCIIT(X)$ and fuzzy commutative ideals as $FCIT(X)$ of BCI-algebra X will be defined and some basic properties of them will be obtained. Next the relation between them and implicative ideals, closed implicative ideals and commutative ideals of BCI-algebras will be obtained such that every $FIIT(X)$ and $FCIIT(X)$ and $FCIT(X)$ will be implicative ideals, closed implicative ideals and commutative ideals of BCI-algebra X, respectively.

Further, the intersection and Cartesian product of them are investigated and we prove that for every $\mu, \nu \in FIIT(X), FCIIT(X), FCIT(X)$ we get that $\mu \cap \nu \in FIIT(X), FCIIT(X), FCIT(X)$. Also, we prove that if

$$\mu \in FIIT(X), FCIIT(X), FCIT(X)$$

and

$$\nu \in FIIT(Y), FCIIT(Y), FCIT(Y),$$

then $\mu \times \nu \in FIIT(X \times Y), FCIIT(X \times Y), FCIT(X \times Y)$, respectively. Finally, under homomorphisms of BCI-algebras, some characterizations of image and preimage of them are obtained such that under homomorphisms of BCI-algebra $\varphi : X \to Y$ if $\mu \in FIIT(X), FCIIT(X), FCIT(X)$ and $\nu \in FIIT(Y), FCIIT(Y), FCIT(Y)$, then $\varphi(\mu) \in FIIT(Y), FCIIT(Y), FCIT(Y)$ and $\varphi^{-1}(\nu) \in FIIT(X), FCIIT(X), FCIT(X)$, respectively.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequal. For more details we refer readers to [**1, 2, 3, 6, 7, 8, 12, 13**].

Definition 2.1. An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(1) $((x * y) * (x * z)) * (z * y) = 0$,

(2) $x * 0 = x$,

(3) $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y, z \in X$. We call the binary operation $*$ on X the $*$ multiolication on X and the constant 0 of the zero element of X. We often write X instead of $(X, *, 0)$ for a BCI-algebra in brevity.

Example 2.2. Let S be a set. Denote 2^S for the power of S in the sense that 2^S is the collection of all subsets of S, $-$ for the set difference and $\emptyset$ for the empty set. Then $(2^S, -, \emptyset)$ is a BCI-algebra.

Example 2.3. Let that $(G, ., e)$ is an Abelian group with e as the unit element. Define a binary operation $*$ on X by putting $x * y = xy^{-1}$. Then $(G, *, e)$ is a BCI-algebra.

Example 2.4. Assume that $(X, \leq)$ is a partially ordered set with the least element 0. Define an operation $*$ on X by

$$x * y = \begin{cases} 0 & \text{if } x \leq y \\ x & \text{if } x \not\leq y \end{cases}$$

then $(X, *, 0)$ is a BCI-algebra.

Proposition 2.1. *An algebra $(X, *, 0)$ of type $(2, 0)$ is a BCI-algebra if and only if it satisfies the following conditions:*
(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$,
for all $x, y, z \in X$.

In a BCI-algebra, we can define a partial ordering " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2. *An algebra $(X, *, 0)$ of type $(2, 0)$ is a BCI-algebra if and only if there is a partial ordering " $\leq$ " on X such that the following conditions hold.*
(1) $(x * y) * (x * z) \leq (z * y)$,
(2) $x * (x * y) \leq y$,
(3) $x * y = 0$ *if and only if* $x \leq y$,
for all $x, y, z \in X$.

Proposition 2.3. *Let $(X, *, 0)$ of type $(2, 0)$ be a BCI-algebra. Then*
(1) $x \leq y$ *implies* $z * y \leq z * x$,
(2) $x \leq y$ *implies* $x * z \leq y * z$,
(3) $(x * y) * z = (x * z) * y$,
(4) $x * (x * (x * y)) = x * y$,
(5) $0 * (x * y) = (0 * x) * (0 * y)$,
for all $x, y, z \in X$.

Definition 2.5. A non-empty subset I of a BCI-algebra X is called an ideal of X if
(1) $0 \in I$,
(2) $x * y \in I$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

An ideal I of a BCI-algebra X is said to be closed if $0 * x \in I$, for all $x \in X$. A non-empty subset I of a BCI-algebra X is said to be an implicative ideal of X if it satisfies:

(1) $0 \in I$, (2) $((x*(x*y))*(y*x))*z \in I$ and $z \in I$ imply $y*(y*x) \in I$, for all $x, y, z \in X$.

A non-empty subset I of a BCI-algebra X is said to be a commutative ideal of X if it satisfies:

(1) $0 \in I$, (2) $(y*(y*(x*(x*y))))*z \in I$ and $z \in I$ imply $x*(x*y) \in I$, for all $x, y, z \in X$.

Definition 2.6. A mapping $f : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ of BCI-algebras is called a BCI-homomorphism if $f(x*y) = f(x)\acute{*}f(y)$, for all $x, y \in X$.
The zero mapping $\theta : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ of BCI-algebras with $\theta(0) = \acute{0}$ is a BCI-homomorphism.

Proposition 2.4. *Let $f : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ of BCI-algebras be a BCI-homomorphism. Then*

(1) $f(0) = \acute{0}$.

(2) f is of the isotonic property in the meaning that $x \leq y$ implies $f(x)\acute{\leq}f(y)$ for all $x, y \in X$.

(3) f is epimorphic if and only if $Im(f) = Y$.

Definition 2.7. Let X be an arbitrary set. A fuzzy subset of X, we mean a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the $[0, 1]$-power set of X and is denoted $[0, 1]^X$. For a fixed $s \in [0, 1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ.

Definition 2.8. Let φ be a function from set X into set Y such that $\mu : X \to [0, 1]$ and $\nu : Y \to [0, 1]$. For all $x \in X, y \in Y$, we define $\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\}$ and $\varphi^{-1}(\nu)(x) = \nu(\varphi(x))$.

Definition 2.9. A t-norm T is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties: (T1) $T(x, 1) = x$ (neutral element),

(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),

(T3) $T(x, y) = T(y, x)$ (commutativity),

(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),

for all $x, y, z \in [0, 1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.10. (1) Standard intersection t-norm $T_m(x, y) = \min\{x, y\}$.

(2) Bounded sum t-norm $T_b(x, y) = \max\{0, x + y - 1\}$.

(3) Algebraic product t-norm $T_p(x, y) = xy$.

(4) Drastic t-norm

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y>1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x=y=0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x,y \in [0,1]$. We say that T be idempotent if for all $x \in [0,1]$ we have $T(x,x) = x$.

Definition 2.11. The function $T_n : \prod_{i=1}[0,1] \to [0,1]$ is defined by

$$T_n(x_1,x_2,...,x_n) = T(x_i, T_{n-1}(x_1,x_2,...,x_{i-1},x_{i+1},...,x_n))$$

for all $1 \leq i \leq n$, where $n \geq 2$ such that $T_2 = T$ and $T_1 = id$ (identity).

Using the induction on n, we have the following two lemmas.

Lemma 2.5. *For every t-norm T and every $x_i, y_i \in [0,1]$, where $1 \leq i \leq n$, and $n \geq 2$, we have*

$$T_n(T(x_1,y_1),T(x_2,y_2),...,T(x_n,y_n)) = T(T_n(x_1,x_2,...,x_n),T_n(y_1,y_2,...,y_n)).$$

Lemma 2.6. *For a t-norm T and every $x_1,x_2,...,x_n \in [0,1]$, where $n \geq 2$, we have*

$$T_n(x_1,x_2,...,x_n) = T(...T(T(T(x_1,x_2),x_3),x_4),x_n) = T(x_1,T(x_2,T(x_3,...T(x_{n-1},x_n)...)))$$

Definition 2.12. Let $\mu, \nu : X \to [0,1]$ and T be a t-norm. We define the intersection of μ and ν as

$$\mu \cap \nu : X \to [0,1]$$

by

$$(\mu \cap \nu)(x) = T(\mu(x), \nu(x))$$

for all $x \in X$.

Let $\mu : X \to [0,1]$ and $\nu : Y \to [0,1]$ and T be a t-norm. The cartesian product of μ and ν is denoted by

$$\mu \times \nu : X \times Y \to [0,1]$$

is defined by

$$(\mu \times \nu)(x,y) = T(\mu(x), \nu(y))$$

for all $(x,y) \in X \times Y$.

Lemma 2.7. *Let T be a t-norm. Then*

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z))$$

for all $x,y,w,z \in [0,1]$.

3. Main Results

Definition 3.1. Let X be a BCI-algebra. Define $\mu : X \to [0,1]$ a fuzzy implicative ideal of X under t-norm T if it satisfies the following inequalities:

(1) $\mu(0) \geq \mu(x)$,

(2) $\mu(y * (y * x)) \geq T(\mu(x * (x * y) * (y * x)), \mu(z))$, for all $x, y, z \in X$.

Denote by $FIIT(X)$, the set of all fuzzy implicative ideals of BCI-algebra X under t-norm T.

Now we get the following example.

Example 3.2. Let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Then $(X, *, 0)$ is a BCI-algebra. Define $\mu : X \to [0,1]$ as

$$\mu(x) = \begin{cases} t_1 & \text{if } x = 0, 1, \\ t_2 & \text{if } x = 2, \end{cases}$$

with $t_1 > t_2$ such that $t_i \in [0,1]$. Let $T(a,b) = T_p(a,b) = ab$ for all $a, b \in [0,1]$ then $\mu \in FIIT(X)$.

Proposition 3.1. *Let $\mu : X \to [0,1]$ and T be idempotent such that $\mu \in FIIT(X)$. Then the set*

$$\mu_s = \{x \in X : \mu(x) \geq s\}$$

is either empty or an implicative ideal of BCI-algebra X for every $s \in [0,1]$.

Proof. As $\mu \in FIIT(X)$ and $\mu_s = \{x \in X : \mu(x) \geq s\}$ be not empty so for any $x \in \mu_s$ then $\mu(x) \geq s$ and $\mu(0) \geq \mu(x) \geq s$ which means that $0 \in \mu_s$.
Let $((x * (x * y)) * (y * x)) * z \in \mu_s$ and $z \in \mu_s$. Thus

$$\mu(y * (y * x)) \geq T(\mu(x * (x * y) * (y * x)), \mu(z)) \geq T(s, s) = s$$

so $y * (y * x) \in \mu_s$. Thus μ_s will be an implicative ideal of BCI-algebra X for every $s \in [0,1]$. □

Example 3.3. Let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Then $(X, *, 0)$ is a BCI-algebra. Define

$$\mu : X \to [0, 1]$$

as

$$\mu(x) = \begin{cases} 0.75 & \text{if } x = 0, \\ 0.65 & \text{if } x = 1, \\ 0.35 & \text{if } x = 2, \end{cases}$$

Let $T(a, b) = T_m(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ then $\mu \in FIIT(X)$. Let $s = 0.50$ then

$$\mu_{0.50} = \{x \in X : \mu(x) \geq 0.50\} = \{0, 1\}$$

with the following Cayley table:

*	0	1
0	0	0
1	1	0

So $\mu_{0.50}$ will be an implicative ideal of BCI-algebra X.

Proposition 3.2. *Let $\mu \in FIIT(X)$ and $\nu \in FIIT(X)$. Then $\mu \cap \nu \in FIIT(X)$.*

Proof. Let $x, y, z \in X$. Then

$$(\mu \cap \nu)(0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x). \tag{1}$$

$$\begin{aligned} &(\mu \cap \nu)(y * (y * x)) = T(\mu(y * (y * x)), \nu(y * (y * x))) \\ &\geq T(T(\mu(x * (x * y) * (y * x)), \mu(z)), T(\nu(x * (x * y) * (y * x)), \nu(z))) \\ &= T(T(\mu(x * (x * y) * (y * x)), \nu(x * (x * y) * (y * x)), T(\mu(z), \nu(z))) \\ &= T((\mu \cap \nu)(x * (x * y) * (y * x), (\mu \cap \nu)(z)). \end{aligned} \tag{2}$$

So

$$(\mu \cap \nu)(y * (y * x)) \geq T((\mu \cap \nu)(x * (x * y) * (y * x), (\mu \cap \nu)(z)).$$

Then $\mu \cap \nu \in FIIT(X)$. □

Example 3.4. Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Then $(X, *, 0)$ is a BCI-algebra. Define

$$\mu : X \to [0, 1]$$

as

$$\mu(x) = \begin{cases} 0.70 & \text{if } x = 0, \\ 0.60 & \text{if } x = 1, \\ 0.50 & \text{if } x = 2, \\ 0.40 & \text{if } x = 3, \end{cases}$$

and

$$\nu : X \to [0, 1]$$

as

$$\nu(x) = \begin{cases} 0.50 & \text{if } x = 0, \\ 0.40 & \text{if } x = 1, \\ 0.30 & \text{if } x = 2, \\ 0.20 & \text{if } x = 3, \end{cases}$$

Let $T(a, b) = T_b(a, b) = \max\{0, a + b - 1\}$ for all $a, b \in [0, 1]$. Then $\mu, \nu \in FIIT(X)$. Also

$$\mu \cap \nu : X \to [0, 1]$$

as

$$\begin{aligned}(\mu \cap \nu)(x) =& T(\mu(x), \nu(x)) = T_b(\mu(x), \nu(x)) \\ =& \max\{0, \mu(x) + \nu(x) - 1\} \\ =& \begin{cases} 0.20 & \text{if } x = 0, \\ 0 & \text{if } x = 1, 2, 3, \end{cases}\end{aligned}$$

So $\mu \cap \nu \in FIIT(X)$.

Proposition 3.3. *Let $\mu \in FIIT(X)$ and $\nu \in FIIT(Y)$. Then $\mu \times \nu \in FIIT(X \times Y)$.*

Proof. Let $(x, y) \in X \times Y$. Then

$$(\mu \times \nu)(0, 0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(y)) = (\mu \times \nu)(x, y).$$

Also let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times Y$. Now, we have

$$\begin{aligned}&(\mu \times \nu)((y_1, y_2) * ((y_1, y_2) * (x_1, x_2))) = (\mu \times \nu)((y_1, y_2) * (y_1 * x_1, y_2 * x_2)) \\ =&(\mu \times \nu)(y_1 * (y_1 * x_1), y_2 * (y_2 * x_2)) \\ =&T(\mu(y_1 * (y_1 * x_1)), \nu(y_2 * (y_2 * x_2))) \\ \geq&T(T(\mu(x_1 * (x_1 * y_1) * (y_1 * x_1)), \mu(z_1)), T(\nu(x_2 * (x_2 * y_2) * (y_2 * x_2)), \nu(z_2))) \\ =&T(T(\mu(x_1 * (x_1 * y_1) * (y_1 * x_1)), \nu(x_2 * (x_2 * y_2) * (y_2 * x_2))), T(\mu(z_1), \nu(z_2))) \\ =&T((\mu \times \nu)(x_1 * (x_1 * y_1) * (y_1 * x_1), x_2 * (x_2 * y_2) * (y_2 * x_2)), (\mu \times \nu)(z_1, z_2)) \\ =&T((\mu \times \nu)((x_1, x_2) * ((x_1, x_2) * (y_1, y_2)) * ((y_1, y_2) * (x_1, x_2))), (\mu \times \nu)(z_1, z_2))\end{aligned}$$

so

$$(\mu \times \nu)((y_1, y_2) * ((y_1, y_2) * (x_1, x_2))) \geq T((\mu \times \nu)((x_1, x_2) * ((x_1, x_2) * (y_1, y_2)) * ((y_1, y_2) * (x_1, x_2))), (\mu \times \nu)(z_1, z_2)).$$

Thus $\mu \times \nu \in FIIT(X \times Y)$. □

Example 3.5. Let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:

$*$	0	1	2
0	0	0	0
1	1	0	2
2	2	0	0

and $Y = \{0, 1, 2\}$ be a set given by the following Cayley table:

$*$	0	1	2
0	0	1	0
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ and $(Y, *, 0)$ will be two BCI-algebras. Define

$$\mu : X \to [0, 1]$$

as

$$\mu(x) = \begin{cases} 0.45 & \text{if } x = 0, \\ 0.25 & \text{if } x = 1, \\ 0.35 & \text{if } x = 2, \end{cases}$$

and

$$\nu : Y \to [0, 1]$$

as

$$\nu(y) = \begin{cases} 0.35 & \text{if } y = 0, \\ 0.20 & \text{if } y = 1, \\ 0.10 & \text{if } y = 2. \end{cases}$$

Let

$$T(a, b) = T_{H_0}(a, b) = \begin{cases} 0 & \text{if } a = b = 0 \\ \frac{ab}{a+b-ab} & \text{otherwise} \end{cases}$$

for all $a, b \in [0, 1]$. Then $\mu \in FIIT(X)$ and $\nu \in FIIT(Y)$. Now

$$\mu \times \nu : X \times Y = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\} \to [0, 1]$$

as

$$\begin{aligned}(\mu\times\nu)(x,y) =&T(\mu(x),\nu(y))\\ =&T_{H_0}(\mu(x),\nu(y))\\ =&\begin{cases}0 & \text{if } \mu(x)=\nu(y)=0\\ \frac{\mu(x)\nu(y)}{\mu(x)+\nu(y)-\mu(x)\nu(y)} & \text{otherwise}\end{cases}\\ =&\begin{cases}0.2451 & \text{if } (x,y)=(0,0)\\ 0.1607 & \text{if } (x,y)=(0,1)\\ 0.0819 & \text{if } (x,y)=(0,2)\\ 0.1707 & \text{if } (x,y)=(1,0)\\ 0.125 & \text{if } (x,y)=(1,1)\\ 0.77 & \text{if } (x,y)=(1,2)\\ 0.2121 & \text{if } (x,y)=(2,0)\\ 0.1459 & \text{if } (x,y)=(2,1)\\ 0.0843 & \text{if } (x,y)=(2,2)\end{cases}\end{aligned}$$

Thus $\mu\times\nu\in FIIT(X\times Y)$.

Proposition 3.4. *Let $\mu_i\in FIIT(X_i)$, where $1\leq i\leq n$, then $\mu=\prod_{i=1}^n\mu_i\in\prod_{i=1}^n X_i=X$ such that*

$$\mu(x)=(\prod_{i=1}^n\mu_i)(x_1,x_2,...,x_n)=T_n(\mu_1(x_1,\mu_2(x_2,...,\mu_n(x_n))$$

for all $x=(x_1,x_2,...,x_n)\in X$.

Proposition 3.5. *If $\mu\in FIIT(X)$ and $\varphi:(X;*,0)\to(Y;\acute{*},\acute{0})$ be an epimorphic BCI-homomorphism of BCI-algebras, then $\varphi(\mu)\in FIIT(Y)$.*

Proof. Let $x\in X$ and $y\in Y$ with $\varphi(x)=y$. Now,

$$\varphi(\mu)(0)=\sup\{\mu(0)\mid 0\in X,\varphi(0)=\acute{0}\}\geq\sup\{\mu(x)\mid x\in X,\varphi(x)=y\}=\varphi(\mu)(y).$$

Also, let $x_i\in X$ and $y_i\in Y$ with $\varphi(x_i)=y_i$ and $i=1,2,3$. Then

$$\begin{aligned}&\varphi(\mu)(y_1\acute{*}(y_1\acute{*}y_2))\\ =&\sup\{\mu(x_1*(x_1*x_2))\mid x_1*(x_1*x_2)\in X,\varphi(x_1*(x_1*x_2))=y_1\acute{*}(y_1\acute{*}y_2)\}\\ \geq&\sup\{T(\mu(x_1*(x_2*x_1)*(x_1*x_2)),\mu(x_3))\mid x_i\in X,\varphi(x_i)=y_i\}\\ =&T(\sup\{\mu_A(x_1*(x_2*x_1)*(x_1*x_2))\mid x_i\in X,\varphi(x_1*(x_2*x_1)*(x_1*x_2))\\ &=y_1\acute{*}(y_2\acute{*}y_1)\acute{*}(y_1\acute{*}y_2)\},\sup\{\mu(x_3)\mid x_3\in X,\varphi(x_3)=y_3\})\\ =&T(\varphi(\mu)(y_1\acute{*}(y_2\acute{*}y_1)\acute{*}(y_1\acute{*}y_2)),\varphi(\mu)(y_3))\end{aligned}$$

Hence,

$$\varphi(\mu)(y_1\acute{*}(y_1\acute{*}y_2))\geq T(\varphi(\mu)(y_1\acute{*}(y_2\acute{*}y_1)\acute{*}(y_1\acute{*}y_2)),\varphi(\mu)(y_3)).$$

Thus $\varphi(\mu) \in FIIT(Y)$. □

Proposition 3.6. *If $\nu \in FIIT(Y)$ and $\varphi : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ be a BCI-homomorphism of BCI-algebras, then $\varphi^{-1}(\nu) \in FIIT(X)$.*

Proof. Let $x \in X$. Then

$$\varphi^{-1}(\nu)(0) = \nu(\varphi(0)) \geq \nu(\varphi(x)) = \varphi^{-1}(\nu)(x).$$

Let $x_1, x_2, x_3 \in X$. Now

$$\begin{aligned}\varphi^{-1}(\nu)(x_1 * (x_1 * x_2))) &= \nu(\varphi(x_1 * (x_1 * x_2)))) \\ &= \nu(\varphi(x_1) * (\varphi(x_1) * \varphi(x_2))) \\ &\geq T(\nu(\varphi(x_1) * (\varphi(x_2) * \varphi(x_1)) * (\varphi(x_1) * \varphi(x_2)), \nu(\varphi(x_3))) \\ &= T(\nu(\varphi(x_1 * (x_2 * x_1) * (x_1 * x_2)), \nu(\varphi(x_3))) \\ &= T(\varphi^{-1}(\nu)(x_1 * (x_2 * x_1) * (x_1 * x_2)), \varphi^{-1}(\nu)(x_3))\end{aligned}$$

then

$$\varphi^{-1}(\nu)(x_1 * (x_1 * x_2))) \geq T(\varphi^{-1}(\nu)(x_1 * (x_2 * x_1) * (x_1 * x_2)), \varphi^{-1}(\nu)(x_3)).$$

Therefore $\varphi^{-1}(\nu) \in FIIT(X)$. □

Example 3.6. Let $X = \{0, 1, 2, 3, 4, 5\}$ and $Y = \{0, 1, 2, 3\}$ be two sets given by the following Cayley tables: and

$*$	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	0	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0.

Then $(X, *, 0)$ and $(Y, *, 0)$ will be two BCI-algebras. Define $\mu : X \to [0, 1]$ as

$$\mu(x) = \begin{cases} 0.6 & \text{if } x = 0, 1, \\ 0.5 & \text{if } x = 2, 3, \\ 0.4 & \text{if } x = 4, \\ 0.3 & \text{if } x = 5. \end{cases}$$

and $\nu : Y \to [0,1]$ as

$$\nu(y) = \begin{cases} 0.55 & \text{if } y = 0, \\ 0.45 & \text{if } y = 1, 2, \\ 0.35 & \text{if } y = 3, \end{cases}$$

Let $T(a,b) = T_p(a,b) = ab$ for all $a, b \in [0,1]$ then $\mu \in FIIT(X)$ and $\nu \in FIIT(Y)$. Define BCI-homomorphism $\varphi : X \to Y$ as

$$\varphi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, 2, 3, \\ 2 & \text{if } x = 4, \\ 3 & \text{if } x = 5, \end{cases}$$

then we get that $\varphi(\mu) : Y \to [0,1]$ as

$$\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\} = \begin{cases} 0.6 & \text{if } y = 0, 1 \\ 0.4 & \text{if } y = 2, \\ 0.3 & \text{if } y = 3, \end{cases}$$

and thus $\varphi(\mu) \in FIIT(Y)$.

Also, we will have that $\varphi^{-1}(\nu) : X \to [0,1]$ as

$$\varphi^{-1}(\nu)(x) = \nu(\varphi(x)) = \begin{cases} 0.55 & \text{if } x = 0, \\ 0.45 & \text{if } x = 1, 2, 3, 4, \\ 0.35 & \text{if } x = 5, \end{cases}$$

therefore $\varphi^{-1}(\nu) \in FIIT(X)$.

Definition 3.7. We say that $\mu : X \to [0,1]$ is a fuzzy closed implicative ideal of BCI-algebra X under t-norm T if it satisfies the following inequalities:

(1) $\mu(0 * x) \geq \mu(x)$,

(2) $\mu(y * (y * x)) \geq T(\mu(x * (x * y) * (y * x)), \mu(z))$, for all $x, y, z \in X$. Denote by $FCIIT(X)$, the set of all fuzzy closed implicative ideals of X under t-norm T.

Example 3.8. Let $X = \{0, 1, 2, 3, 4\}$ be a set given by the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then $(X, *, 0)$ is a BCI-algebra. Define $\mu : X \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.65 & \text{if } x = 0, \\ 0.45 & \text{if } x = 1, \\ 0.30 & \text{if } x = 2, \\ 0.55 & \text{if } x = 3, \\ 0.35 & \text{if } x = 4, \end{cases}$$

Let $T(a,b) = T_p(a,b) = ab$ for all $a, b \in [0,1]$ then $\mu \in FCIIT(X)$.

Proposition 3.7. *Let $\mu : X \to [0,1]$ and T be idempotent and $\mu \in FCIIT(X)$. Then*

$$\mu_s = \{x \in X : \mu(x) \geq s\}$$

is either empty or a closed implicative ideal of BCI-algebra X for every $s \in [0,1]$.

Proof. Let $\mu \in FCIIT(X)$ and $\mu_s = \{x \in X : \mu(x) \geq s\}$ be not empty. Thus for any $x \in \mu_s$ we get $\mu(x) \geq s$ and so $\mu(0 * x) \geq \mu(x) \geq s$ and thus $\mu(0 * x) \geq s$ which means that $0 * x \in \mu_s$.

Also, let $x * (x * y) * (y * x) \in \mu_s$ and $z \in \mu_s$. Then

$$\mu(y * (y * x)) \geq T(\mu(x * (x * y) * (y * x)), \mu(z)) \geq T(s, s) = s$$

thus $y*(y*x) \in \mu_s$. Then μ_s is a closed implicative ideal of X for every $s \in [0,1]$. □

Example 3.9. Let $X = \{0, 1, 2, 3, 4\}$ be a set given by the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCI-algebra. Define

$$\mu : X \to [0,1]$$

as

$$\mu(x) = \begin{cases} 0.95 & \text{if } x = 0, \\ 0.85 & \text{if } x = 1, \\ 0.75 & \text{if } x = 2, \\ 0.65 & \text{if } x = 3, \\ 0.55 & \text{if } x = 4, \end{cases}$$

Let $T(a,b) = T_m(a,b) = \min\{a, b\}$ for all $a, b \in [0,1]$ then $\mu \in FCIIT(X)$. Let $s = 0.60$ then

$$\mu_{0.60} = \{x \in X : \mu(x) \geq 0.60\} = \{0, 1, 2, 3\}$$

with the following Cayley table: thus $\mu_{0.60}$ will be a closed implicative ideal of

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

BCI-algebra X.

As results about fuzzy implicative ideals of BCI-algebra X under t-norm T, we get the following propositions and we omit the proofs of them.

Proposition 3.8. *(1) Let* $\mu, \nu \in FCIIT(X)$*. Then* $\mu \cap \nu \in FCIIT(X)$*.*
(2) Let $\mu \in FCIIT(X)$ *and* $\nu \in FCIIT(Y)$*. Then* $\mu \times \nu \in FCIIT(X \times Y)$*.*

Proposition 3.9. *If* $\mu \in FCIIT(X)$ *and* $\varphi : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ *be an epimorphic BCI-homomorphism of BCI-algebras, then* $\varphi(\mu) \in FCIIT(Y)$*.*

Proposition 3.10. *If* $\nu \in FCIIT(Y)$ *and* $\varphi : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ *be a BCI-homomorphism of BCI-algebras, then* $\varphi^{-1}(\nu) \in FCIIT(X)$*.*

Definition 3.10. Define $\mu : X \to [0, 1]$ is a fuzzy commutative ideal of BCI-algebra X under t-norm T if it satisfies the following inequalities:
(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x * (x * y)) \geq T(\mu(y * (y * (x * (x * y)))), \mu(z))$, for all $x, y, z \in X$.
Denote by $FCIT(X)$, the set of all fuzzy commutative ideals of BCI-algebra X under t-norm T.

Example 3.11. Let $X = \{0, 1, 2, 3, 4\}$ be a set given by the following Cayley table:

$*$	0	1	2	3	4
0	0	0	4	3	2
1	1	0	4	3	2
2	2	2	0	4	3
3	3	3	2	0	4
4	4	4	3	2	0

Then $(X, *, 0)$ is a BCI-algebra. Define $\mu : X \to [0, 1]$ as

$$\mu(x) = \begin{cases} 0.55 & \text{if } x = 0, 3, \\ 0.25 & \text{if } x = 1, 2, \\ 0.15 & \text{if } x = 4. \end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ for all $a, b \in [0, 1]$ then $\mu \in FCIT(X)$.

Proposition 3.11. *Let $\mu : X \to [0,1]$ and T be idempotent. If $\mu \in FCIT(X)$, then*

$$\mu_s = \{x \in X : \mu(x) \geq s\}$$

is either empty or a commutative ideal of BCI-algebra X for every $s \in [0,1]$.

PROOF. Let $\mu \in FCIT(X)$ and $\mu_s = \{x \in X : \mu(x) \geq s\}$ be not empty. Then for any $x \in \mu_s$ we have $\mu(x) \geq s$ and so $\mu(0) \geq \mu(x) \geq s$ and which means that $0 \in \mu_s$. Also, let $y * (y * (x * (x * y))) \in \mu_s$ and $z \in \mu_s$. Then

$$\mu(x * (x * y)) \geq T(\mu(y * (y * (x * (x * y)))), \mu(z)) \geq T(s,s) = s,$$

thus $x * (x * y) \in \mu_s$. Then μ_s is a commutative ideal of X for every $s \in [0,1]$. □

Example 3.12. Let $X = \{0, a, b, c, d\}$ be a set given by the following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	d	a	0

Then $(X, *, 0)$ is a BCI-algebra. Define

$$\mu : X \to [0,1]$$

as

$$\mu(x) = \begin{cases} 0.55 & \text{if } x = 0, \\ 0.45 & \text{if } x = a, \\ 0.35 & \text{if } x = b, \\ 0.25 & \text{if } x = c, \\ 0.15 & \text{if } x = d, \end{cases}$$

and will be Cayley table for $s = 0.20$ with $\mu_{0.20} = \{x \in X : \mu(x) \geq 0.20\} =$

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

$\{0, a, b, c\}$. Let $T(x,y) = T_m(x,y) = \min\{x,y\}$ for all $x, y \in [0,1]$ then $\mu \in FCIIT(X)$. Thus $\mu_{0.20}$ will be a commutative ideal of BCI-algebra X.

Proposition 3.12. *Let $\mu \in FCIT(X)$ and $\nu \in FCIT(Y)$. Then $\mu \times \nu \in FCIT(X \times Y)$.*

PROOF. Let $(x,y)\in X\times Y$. Then

$$(\mu\times\nu)(0,0)=T(\mu(0),\nu(0))\geq T(\mu(x),\nu(y))=(\mu\times\nu)(x,y).$$

Also let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times Y$. Now

$$\begin{aligned}
&(\mu\times\nu)((x_1,x_2)*((x_1,x_2)*(y_1,y_2)))=(\mu\times\nu)((x_1,x_2)*(x_1*y_1,x_2*y_2))\\
=&(\mu\times\nu)(x_1*(x_1*y_1),x_2*(x_2*y_2))=T(\mu(x_1*(x_1*y_1)),\nu(x_2*(x_2*y_2)))\\
\geq&T(T(\mu(y_1*(y_1*(x_1*(x_1*y_1)))),\mu(z_1)),T(\nu(y_2*(y_2*(x_2*(x_2*y_2)))),\nu(z_2)))\\
=&T(T(\mu(y_1*(y_1*(x_1*(x_1*y_1)))),\nu(y_2*(y_2*(x_2*(x_2*y_2))))),T(\mu(z_1),\mu_B(z_2)))\\
=&T((\mu\times\nu)(y_1*(y_1*(x_1*(x_1*y_1))),y_2*(y_2*(x_2*(x_2*y_2)))),(\mu\times\nu)(z_1,z_2))\\
=&T((\mu\times\nu)((y_1,y_2)*((y_1,y_2)*((x_1,x_2)*((x_1,x_2)*(y_1,y_2))))),(\mu\times\nu)(z_1,z_2)).
\end{aligned}$$

Thus

$$\begin{aligned}
&(\mu\times\nu)((x_1,x_2)*((x_1,x_2)*(y_1,y_2)))\\
\geq&T(\mu\times\nu)((y_1,y_2)*((y_1,y_2)*((x_1,x_2)*((x_1,x_2)*(y_1,y_2))))),(\mu\times\nu)(z_1,z_2)).
\end{aligned}$$

Therefore $\mu\times\nu\in FCIT(X\times Y)$. □

Example 3.13. Let $X=\{0,a,b\}$ be a set given by the following Cayley table:

*	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

and $Y=\{0,a,b\}$ be a set given by the following Cayley table:

*	0	a	b
0	0	0	b
a	a	0	a
b	b	b	0

Then $(X,*,0)$ and $(Y,*,0)$ will be two BCI-algebras. Define

$$\mu:X\to[0,1]$$

as

$$\mu(x)=\begin{cases}0.4 & \text{if } x=0,\\ 0.2 & \text{if } x=a,\\ 0.3 & \text{if } x=b,\end{cases}$$

and

$$\nu:Y\to[0,1]$$

as

$$\nu(y) = \begin{cases} 0.3 & \text{if } y = 0, \\ 0.25 & \text{if } y = a, \\ 0.15 & \text{if } y = b. \end{cases}$$

Let $T(m,n) = T_m(m,n) = \min\{m,n\}$ for all $m,n \in [0,1]$. Then $\mu \in FIIT(X)$ and $\nu \in FIIT(Y)$. Also

$$\mu \times \nu : X \times Y = \{(0,0),(0,a),(0,b),(a,0),(a,a),(a,b),(b,0),(b,a),(b,b)\} \to [0,1]$$

as

$$(\mu \times \nu)(x,y) = T(\mu(x),\nu(y)) = \min\{\mu(x),\nu(y)\} = \begin{cases} 0.3 & \text{if } (x,y) = (0,0), \\ 0.25 & \text{if } (x,y) = (0,a), \\ 0.15 & \text{if } (x,y) = (0,b), \\ 0.2 & \text{if } (x,y) = (a,0), \\ 0.2 & \text{if } (x,y) = (a,a), \\ 0.15 & \text{if } (x,y) = (a,b), \\ 0.3 & \text{if } (x,y) = (b,0), \\ 0.25 & \text{if } (x,y) = (b,a), \\ 0.15 & \text{if } (x,y) = (b,b), \end{cases}$$

Therefore $\mu \times \nu \in FIIT(X \times Y)$.

Proposition 3.13. *Let $\mu, \nu \in FCIT(X)$. Then $\mu \cap \nu \in FCIT(X)$.*

Proof. Let $x, y, z \in X$. Then

$$(\mu \cap \nu)(0) = T(\mu(0),\nu(0)) \geq T(\mu(x),\nu(x)) = (\mu \cap \nu)(x), \tag{3}$$

and

$$\begin{aligned} &(\mu \cap \nu)(x*(x*y)) = T(\mu(x*(x*y)),\nu(x*(x*y))) \\ &\geq T(T(\mu(y*(y*(x*(x*y)))),\mu(z)),T(\nu(y*(y*(x*(x*y)))),\nu(z))) \\ &= T(T(\mu(y*(y*(x*(x*y)))),\nu(y*(y*(x*(x*y))))),T(\mu(z),\nu(z))) \\ &= T((\mu \cap \nu)(y*(y*(x*(x*y)))),(\mu \cap \nu)(z)) \end{aligned} \tag{4}$$

so

$$(\mu \cap \nu)(x*(x*y)) \geq T((\mu \cap \nu)(y*(y*(x*(x*y)))),(\mu \cap \nu)(z)).$$

Now (3) and (4) give us that $\mu \cap \nu \in FCIT(X)$. □

Example 3.14. Let $X = \{0,a,b,c\}$ be a set given by the following Cayley table:

Then $(X,*,0)$ is a BCI-algebra. Define

$$\mu, \nu : X \to [0,1]$$

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	c
c	c	c	c	0

as

$$\mu(x) = \begin{cases} 0.7 & \text{if } x = 0, \\ 0.6 & \text{if } x = a, \\ 0.5 & \text{if } x = b, \\ 0.4 & \text{if } x = c, \end{cases}$$

and

$$\nu(x) = \begin{cases} 0.55 & \text{if } x = 0, \\ 0.45 & \text{if } x = a, \\ 0.35 & \text{if } x = b, \\ 0.25 & \text{if } x = c, \end{cases}$$

Let $T(a,b) = T_p(a,b) = ab$ for all $a, b \in [0,1]$. Then $\mu, \nu \in FCIIT(X)$. Also

$$(\mu \cap \nu)(x) = T(\mu(x), \nu(x)) = \mu(x)\nu(x) = \begin{cases} 0.385 & \text{if } x = 0, \\ 0.27 & \text{if } x = a, \\ 0.175 & \text{if } x = b, \\ 0.1 & \text{if } x = c, \end{cases}$$

thus $\mu \cap \nu \in FCIT(X)$.

Proposition 3.14. *If $\mu \in FCIT(X)$ and $\varphi : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ be an epimorphic BCI-homomorphism of BCI-algebras, then $\varphi(\mu) \in FCIT(Y)$.*

Proof. Let $x \in X$ and $y \in Y$ with $\varphi(x) = y$. Now

$$\varphi(\mu)(0) = \sup\{\mu(0) \mid 0 \in X, \varphi(0) = \acute{0}\} \geq \sup\{\mu(x) \mid x \in X, \varphi(x) = y\} = \varphi(\mu)(y).$$

Also let $x_i \in X$ and $y_i \in Y$ with $\varphi(x_i) = y_i$ and $i = 1, 2, 3$. Then

$$\begin{aligned} &\varphi(\mu)(y_1 \acute{*} (y_1 \acute{*} y_2)) \\ =& \sup\{\mu(x_1 * (x_1 * x_2)) \mid x_1 * (x_1 * x_2) \in X, \varphi(x_1 * (x_1 * x_2)) = y_1 \acute{*} (y_1 \acute{*} y_2)\} \\ \geq& \sup\{T(\mu(x_2 * (x_2 * (x_1 * (x_1 * x_2)))), \mu(x_3)) \mid x_i \in X, \varphi(x_i) = y_i\} \\ =& T\left(\sup\left\{\mu(x_2 * (x_2 * (x_1 * (x_1 * x_2)))) \mid x_i \in X, \varphi(x_2 * (x_2 * (x_1 * (x_1 * x_2))))\right.\right. \\ &\left.\left. = y_2 \acute{*} (y_2 \acute{*} (y_1 \acute{*} (y_1 \acute{*} y_2)))\right\}, \sup\{\mu(x_3) \mid x_3 \in X, \varphi(x_3) y_3\}\right) \\ =& T(\varphi(\mu)(y_2 \acute{*} (y_2 \acute{*} (y_1 \acute{*} (y_1 \acute{*} y_2)))), \varphi(\mu)(y_3)), \end{aligned}$$

thus

$$\varphi(\mu)(y_1 \acute{*} (y_1 \acute{*} y_2)) \geq T(\varphi(\mu)(y_2 \acute{*} (y_2 \acute{*} (y_1 \acute{*} (y_1 \acute{*} y_2)))), \varphi(\mu) = (y_3)).$$

Therefore $\varphi(\mu) \in FCIT(Y)$. $\square$

Proposition 3.15. *If $\nu \in FCIT(Y)$ and $\varphi : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ be a BCI-homomorphism of BCI-algebras, then $\varphi^{-1}(\nu) \in FCIT(X)$.*

Proof. Let $x \in X$. Then

$$\varphi^{-1}(\nu)(0) = \nu(\varphi(0)) \geq \nu(\varphi(x)) = \varphi^{-1}(\nu)(x).$$

Let $x_1, x_2, x_3 \in X$. Now

$$\begin{aligned}\varphi^{-1}(\nu)(x_1 * (x_1 * x_2))) &= \nu(\varphi(x_1 * (x_1 * x_2)))) \\ &= \nu(\varphi(x_1) * (\varphi(x_1) * \varphi(x_2))) \\ &\geq T(\nu(\varphi(x_2) * (\varphi(x_2) * (\varphi(x_1) * (\varphi(x_1) * \varphi(x_2))))), \nu(\varphi(x_3))) \\ &= T(\nu(\varphi(x_2 * (x_2 * (x_1 * (x_1 * x_2)))), \nu(\varphi(x_3))) \\ &= T(\varphi^{-1}(\nu)(x_2 * (x_2 * (x_1 * (x_1 * x_2))), \varphi^{-1}(\nu)(x_3))\end{aligned}$$

then

$$\varphi^{-1}(\nu)(x_1 * (x_1 * x_2))) \geq T(\varphi^{-1}(\nu)(x_2 * (x_2 * (x_1 * (x_1 * x_2))), \varphi^{-1}(\nu)(x_3)).$$

Therefore $\varphi^{-1}(\nu) \in FCIT(X)$. □

Example 3.15. Let $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1, 2, 3\}$ be two sets given by the following Cayley tables:

$*$	0	1	2	3	4
0	0	0	2	3	3
1	1	0	1	4	3
2	2	2	0	3	3
3	3	3	3	0	0
4	4	3	4	1	0

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, *, 0)$ and $(Y, *, 0)$ will be two BCI-algebras. Define $\mu : X \to [0, 1]$ as

$$\mu(x) = \begin{cases} 0.25 & \text{if } x = 0, \\ 0.15 & \text{if } x = 1, 2, \\ 0.1 & \text{if } x = 3, 4, \end{cases}$$

and $\nu : Y \to [0, 1]$ as

$$\nu(y) = \begin{cases} 0.75 & \text{if } y = 0, 1, \\ 0.55 & \text{if } y = 2, \\ 0.45 & \text{if } y = 3, \end{cases}$$

Let $T(a,b) = T_m(a,b) = \min\{a,b\}$ for all $a, b \in [0,1]$ then $\mu \in FCIT(X)$ and $\nu \in FCIT(Y)$. Define BCI-homomorphism $\varphi : X \to Y$ as

$$\varphi(x) = \begin{cases} 0 & \text{if } x = 0, 1, \\ 1 & \text{if } x = 2, \\ 2 & \text{if } x = 3, \\ 3 & \text{if } x = 4. \end{cases}$$

Then we have that $\varphi(\mu) : Y \to [0,1]$ as

$$\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\} = \begin{cases} 0.25 & \text{if } y = 0, \\ 0.15 & \text{if } y = 1, \\ 0.1 & \text{if } y = 2, 3. \end{cases}$$

Thus $\varphi(\mu) \in FCIT(Y)$. Also we will get that $\varphi^{-1}(\nu) : X \to [0,1]$ as

$$\varphi^{-1}(\nu)(x) = \nu(\varphi(x)) = \begin{cases} 0.75 & \text{if } x = 0, 1, 2, \\ 0.55 & \text{if } x = 3, \\ 0.45 & \text{if } x = 4, \end{cases}$$

then $\varphi^{-1}(\nu) \in FCIT(X)$.

4. Conclusions

In this paper, we introduced the concept of fuzzy implicative ideals, fuzzy closed implicative ideals and fuzzy commutative ideals of BCI-algebras with respect to T-norms and some basic properties are obtained. Also, the author investigated the intersection and Cartesian product of them and some basic properties are obtained. Finally, as important results, we investigated them under homomorphisms. Now One can investigate fuzzy ideals of BCH-algebras by using t-norms as we considered and this can be an open problem.

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