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Parameterised eight order iterative structures requiring no function derivative for solving nonlinear equation

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Abstract. In this work, the function derivatives in the double Newton iterative structure were annihilated by the use of function estimation with polynomial interpolation and divided difference operator. This resulted in the development of a modified double Newton iterative structure that requires no function derivative. To enhance the modified double Newton iterative structure, it was composed with an iterative structure that involves weight functions and requires an additional function evaluation to produce two parameterized families of iterative structures with convergence order eight. The conditions for convergence of the developed iterative structures were established via the Taylor series approach. The applicability of the developed iterative structures was tested on some nonlinear equations and from the obtained computational results, they are highly competitive when compared with some good existing iterative structures of the same order of convergence.

1. Introduction

There are plethora of Iterative Structures (IS) for determining the simple solution σ of nonlinear (NL) equation $w(s) = 0$, that require functions derivatives evaluation in their structures. Some standard examples of this kind of IS can be found in the literature [1, 2, 3, 4, 5, 6, 7] and reference there in. The most famous of them all, is the Newton IS (NIS) put forward in [7]. The NIS require the evaluation of one

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function and its derivative in an iteration cycle. It is expressed as

$$s_{k+1} = s_k - \frac{w(s_k)}{w'(s_k)}, k = 0, 1, 2, \cdots . \tag{1}$$

In fact, most existing IS for obtaining the zero of NL equations are variants of the NIS and are either two-point or multipoint in structure. The presence of evaluation of functions derivatives in an IS, will incur more computation cost. Furthermore, in some cases, obtaining derivatives of functions can be daunting. To circumvent this setback, the functions derivatives that appear in an IS are usually annihilated by means of estimation. Several successful attempts have been made by researchers to convert some of the existing IS that require functions derivative to IS that are derivative free. One earliest and famous of these attempts is the one due to Steffensen presented in [8]. It is a modification of the IS in (1) and put forward as:

$$s_{k+1} = s_k - \frac{w(s_k)}{w[s_k, \beta_k]}, k = 0, 1, 2, \cdots , \tag{2}$$

where $w[\cdot, \cdot]$ is a divided difference operator. In the IS (1), the derivative $w'(s_k)$ was estimated by a divided difference $w[s_k, \beta_k]$, where $\beta_k = s_k + w(s_k)$. It is important to note that the IS (2) preserved the convergence order (CO) property of (1) and so, the IS in (2) is regarded as a tough competitor to the NIS in (1).

Since the advent of the concept used in (2), many authors have employed it together with weight function technique to further improve the IS (1). These improvements were usually put forward as two-point or multipoint IS utilizing (2) or its variants as the predictor iterative function, see [**9, 10, 11, 12, 13, 14**] and some references in them.

The fundamental goal of this work is to put forward an efficient families of IS that is designed via the estimation of function derivative in the double NIS with polynomial and the use of weight functions. Consequently, two parameterized families of derivative free IS for determining the solution of NL equation is developed.

The manuscript structure has Section 2 dedicated to the IS development, while Section 3 contains the convergence analysis of the developed IS. The numerical implementation of the IS is provided in Section 4. The concluding remarks were given in Section 5.

2. Development of the IS

We begin by acknowledging the double Newton IS given as:

$$\begin{aligned} y_k &= s_k - \frac{w(s_k)}{w'(s_k)}, \\ s_{k+1} &= y_k - \frac{w(y_k)}{w'(y_k)}. \end{aligned} \tag{3}$$

The IS in (3) require the evaluation of two functions $w(\cdot)$ and two functions derivatives $w'(\cdot)$. To enhance the IS (3), a strategy is employed targeted at annihilating the functions derivatives $w'(\cdot)$ in the first and second step. To this end, an estimation of the functions derivatives $w'(\cdot)$ in the IS is made by considering the interpolation polynomial as following:

$$\begin{aligned} I\left(x\right) &= \sum_{i=0}^{2} a_i \left(x - s_k\right)^i, \\ I'\left(x\right) &= a_1 + 2a_2 \left(x - s_k\right). \end{aligned} \tag{4}$$

For some known values substituted in (4), such as

$$\begin{aligned} I\left(y_k\right) = w(y_k) &= \sum_{i=0}^{2} a_i \left(y_k - s_k\right)^i, \\ I'\left(y_k\right) = w'\left(y_k\right) &= a_1 + 2a_2 \left(y_k - s_k\right) \end{aligned} \tag{5}$$

where $w\left(s_k\right) = a_0$, $w'\left(s_k\right) = a_1$, and $w''(s_k) = a_2$, the equations in (5) can be solved to obtain

$$w'\left(y_k\right) \approx 2\left[\frac{w(y_k) - w(s_k)}{y_k - s_k}\right] - w'(s_k). \tag{6}$$

By the estimation of the derivative $w'(s_k)$, (6) can be rewritten as:

$$w'\left(y_k\right) \approx 2w\left[s_k, y_k\right] - w\left[s_k, \eta_k\right], \tag{7}$$

where $\eta_k = s_k + \alpha\left(w(s_k)\right)^m$, $m \geq 2$, $\alpha \in \Re - \{0\}$, and $w\left[\cdot, \cdot\right]$ is a divided difference operator. Motivated by (7), a modified double Newton iterative structure (MDNIS) which is a derivative free version of (3) is obtained and put forward as:

$$\begin{aligned} y_k &= s_k - \frac{w\left(s_k\right)}{w\left[s_k, \eta_k\right]}; \\ s_{k+1} &= y_k - \frac{w\left(y_k\right)}{2w\left[s_k, y_k\right] - w\left[s_k, \eta_k\right]}. \end{aligned} \tag{8}$$

Observe that, the MDNIS will require the evaluation of three distinct functions in one complete iteration cycle and is derivative free. It was proven in Yasmin et al., [**13**] that for $m = 2$, the IS in (8) is of CO four. Consequently, it is optimal as conjectured by Kung and Traub [**15**] and possesses better efficiency than the NIS and double Newton IS presented in (1) and (3) respectively.

In order to improve the CO and efficiency of the IS (8), we compound it with an iterative step that involves three real-valued weight functions (RVWF) $M(v)$, $P(t)$

and $G(s)$ as following:

$$\begin{aligned} y_k &= s_k - \frac{w\left(s_k\right)}{w\left[s_k, \eta_k\right]}; \\ z_k &= y_k - \frac{w\left(y_k\right)}{2w\left[s_k, y_k\right] - w\left[s_k, \eta_k\right]} \\ s_{k+1} &= z_k - \frac{w\left(z_k\right)}{2w\left[s_k, y_k\right] - w\left[s_k, \eta_k\right]}\left[M(v) \times P(t) \times G(u)\right] \end{aligned} \tag{9}$$

where $v = \frac{w(y_k)}{w(s_k)}$, $t = \frac{w(z_k)}{w(\eta_k)}$, $u = \frac{w(z_k)}{w(y_k)}$ and the RVWF with their corresponding Taylor series expression about 0 as

$$M(v) = M(0) + \sum_{i=1}^{8} \frac{1}{i!} M^{(i)}(0)(v)^i, \tag{10}$$

$$P(t) = P(0) + \sum_{i=1}^{8} \frac{1}{i!} P^{(i)}(0)(t)^i, \tag{11}$$

$$G(u) = G(0) + \sum_{i=1}^{8} \frac{1}{i!} G^{(i)}(0)(u)^i, \tag{12}$$

with $M^{(i)}(0)$, $P^{(i)}(0)$ and $G^{(i)}(0)$ to implies, the ith derivative of the functions M, P and G and evaluated at 0. Our claim in this case, the IS (8) is of CO eight for some conditions placed on the parameters $M^{(i)}(0)$, $P^{(i)}(0)$ and $G^{(i)}(0)$, for $1 \leq i \leq 4$. The proof of this claim is established in the proof of Theorem 3.1.

To reduce the number of weight functions used in the IS (9), the third step of the IS is replaced with a new iterative function given as

$$s_{k+1} = z_k - \frac{w(z_k)}{w\left[y_k, z_k\right]}\left[M(v) \times P(t)\right] \tag{13}$$

In this case, our claim is that the IS (13) is of CO eight for some suitable conditions placed on $M^{(i)}(0)$ and $P^{(i)}(0)$, for $1 \leq i \leq 4$. This will be established in the proof of Theorem 3.2.

3. Convergence Analysis of the IS

The convergence of sequence of approximations of the solution of NL equation, produced by the IS (9) and (13) are established in this section. In establishing the convergence, it is important to note that if $e_k = s_k - \sigma$ is an IS error at kth iteration and that equation of the form $e_{k+1} = \Omega e_k^{\nu} + O(e_k^{\nu+1})$ can be derived from the IS via the method of Taylor series expansions of the functions $w(\cdot)$ and $w'(\cdot)$, then e_{k+1} is referred to as error equation, Ω is the error constant and ν is CO of the IS. Furthermore, suppose the error equation holds for an IS as described above, then the IS Efficiency index (E_{eff}) is measured as $E_{eff} = \nu^{\frac{1}{T}}$, T is number of all distinct functions $w(\cdot)$ in the IS cycle.

The convergence of the IS (9) and (13) is considered in the proof of Theorem 3.1 and Theorem 3.2.

Theorem 3.1. *Consider the scalar function $w : D \subset \Re \to \Re$ that is differentiable in D and has a simple solution $\sigma \in D$. If s_0 is close to σ, then the sequence $\{s_k\}_{k\geq 0}, (s_k \in D)$ of approximations generated by the IS in (9) will converge to σ with CO eight when $M(v)$, $P(t)$and $G(u)$ are jointly subjected to the conditions $M(0) = P(0) = G(0) = 1, M'(0) = 0, M''(0) = 2$, $M'''(0) = 12$, $M^{(iv)}(0) < \infty$, $P'(0) = 2$, $P''(0) = 2$, $G'(0) = 1$ and $G''(0) < \infty$.*

Proof. Let $c_k = \frac{w^{(i)}(\sigma)}{i!w'(\sigma)}, i \geq 2$ and set s=s_k in the Taylor's series expansion of $w(s)$ and $w'(s)$, then the expansion for $w(s_k)$ and $w(\eta_k)$ can be obtained as

$$w(s_k) = w'(\sigma)\left[e_k + \sum_{n=2}^{9} c_n e_k^n + O\left(e_k^{10}\right)\right], k\text{=}0, 1, 2, \cdots \tag{14}$$

and

$$\begin{aligned} w(\eta_k) = w'(\sigma)\big[&e_k + c_2e_k^2 + (c_3+\alpha)e_k^3 + (c_4+5\alpha c_2)c_k^4 \\ &+ (c_5 + 6\alpha c_3 + 9\alpha c_2^2)e_k^5 + (c_6 + 7\alpha(3c_2c_3 + c_2^3) + \alpha^2 c_2)e_k^6 + \cdots \\ &+(3\alpha^2(5c_2^3 + 8c_2c_3 + 2c_4) + 9\alpha(c_2^3c_3 + \cdots + 3c_2(c_2^3 + c_5) + c_6) + c_8)e_k^8 + O(e_k^8)\big] \end{aligned} \tag{15}$$

respectively. By using (14) and (15), we get

$$\begin{aligned} w[s_k, \eta_k] =&\frac{w(\eta_k) - w(s_k)}{\eta_k - s_k} = 1 + 2c_2e_k + 3c_3e_k^2 + (ac_2 + 4c_4)e_k^3 \\ &+ (3\alpha(c_2^2 + c_3) + 5c_5)e_k^4 + 3(\alpha(c_2^3 + 4c_2c_3 + 2c_4) + 2c_6)e_k^5 \\ &+ (\alpha^2c_3 + \alpha(c_2^4 + 15c_2^2 + 9c_3^2 + 21c_2c_4 + 10c_5) + 7c_7)e_k^6 + O(e_k^6). \end{aligned} \tag{16}$$

The expressions in (14) and (16), can be used to get

$$\begin{aligned} y_k =&s_k - \frac{w(s_k)}{w[s_k, \eta_k]} = \alpha + c_2e_k^2 + (2c_3 - 2c_2^2)e_k^3 + (3c_4 - 7c_2c_3 + 4c_2^3 + \alpha c_2)e_k^4 \\ &+ (4c_5 - 10c_2c_4 - 6c_3^2 + 20c_2^2c_3 + 3\alpha c_3 - 8c_3^4)e_k^5 \\ &+ (5c_6 - 13c_2c_5 + 2\alpha(3c_4 - c_2c_3 + c_2^3) + 16c_2^5 - \cdots + 28c_2^2c_4 - 17c_3c_4)e_k^6 + O(e_k^7). \end{aligned} \tag{17}$$

Equation (17) is then used to get the Taylor's expansion of $w(y_k)$ as

$$\begin{aligned} w(y_k) =&w'(\sigma)[c_2e_k^2 + (2c_3 - 2c_2^2)e_k^3 + (3c_4 - 7c_2c_3 + 5c_2^3 + \alpha c_2)e_k^4 \\ &+ (4c_5 - 10c_2c_4 + (2c_2^2(2c_3 - 2c_2^2) - 6c_3^2 + 20c_2^2 + 3\alpha c_3 - 8c_2^4))e_k^5 \\ &+ (16c_5^2 - 51c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 + 2\alpha(c_2^3 - c_2c_3 + 3c_4) \\ &+ c_2((-2c_2^2 + 2c_3)^2 + 2c_2(\alpha c_2 + 14c_3^2 - 7c_2c_3 + 3c_4)))e_k^6 + O(e_k^7)]. \end{aligned} \tag{18}$$

Using (14) and (18), we have

$$\begin{aligned} w\left[s_k, y_k\right] =& \frac{w(y_k) - w(s_k)}{y_k - s_k} = 1 + c_2 e_k + \left(c_2^2 - c_3\right) e_k^2 + (c_4 - 2c_2^3 + 3c_2c_3)e_2^3 \\ &+ (c_5 + 4c_2^4 - 8c_2^2c_3 + 2c_3^2 + 4c_2c_4 + \alpha c_2^2)e_k^4 \\ &+ (-8c_2^2 + 4\alpha c_2c_3 + 20c_2^3c_3 + \cdots + 5c_2c_5)e_k^5 + O(e_k^6). \end{aligned} \tag{19}$$

By the application of the expressions in (16)-(19), the following expansion for z_k in (9) is obtained as:

$$\begin{aligned} z_k =& \sigma + (c_2^3 - c_2c_3)e_k^4 - (\alpha c_2^2 + 2(c_2c_4 + c_3^2 - 4c_2^2c_3 + 2c_2^4))e_k^5 \\ &+ (10c_2^5 - 30c_2^3c_3 + 18c_2c_3^2 + \alpha(c_2^3 - 6c_2c_3) - 3c_2c_5 - 7c_3c_4 + 12c_2^2c_4)e_k^6 + O(e_k^7). \end{aligned} \tag{20}$$

Now;

$$\begin{aligned} w(z_k) =& w'(\sigma)\left[(c_2^3 - c_2c_3)e_k^4 - ((c_2c_4 + c_3^2 - 4c_2^2 + 2c_2^4) + \alpha c_2^2)e_k^5\right. \\ &+ (10c_2^5 - 30c_2^3c_3 + 18c_2c_3^2 + 12c_2^2c_4 - 7c_3c_4 - 3c_2c_5 + \alpha(c_2^3 - 6c_2c3))e_k^6 \\ &\left.+ O(e_k^7)\right], \end{aligned} \tag{21}$$

$$\begin{aligned} v =& \frac{w(y_k)}{w(s_k)} = c_2e_k + (-3c_2^2 + 2c_3)e_k^2 + (\alpha c_2 + 8c_2^3 - 10c_2c_3 + 3c_4)e_k^3 \\ &+ (-\alpha c_2^2 - 20c_2^4 + 3\alpha c_3 + 37c_2^2c_3 - 8c_3^2 - 14c_2c_4 + 4c_5)e_k^4 \\ &+ (48c_2^5 - 118c_2^3c_3 + 55c_2c_3^2 + \cdots + \alpha(5c_2^3 - 6c_2c_3 + 6c_4) - 18c_2c_5 + 5c_6)e_k^5 \\ &+ (-112c_2^6 + 344c_2^4c_3 + 26c_3^3 + \alpha^2(c_3 - c_2^2) + \cdots + 2c_2(75c_3c_4 - 11c_6) + 6c_7)e_k^6 \\ &+ (256c_2^7 - 944c_2^5c_3 + \cdots + c_2^2(-693c_3c_4 + 79c_6 + \cdots + \cdots 7c_8))e_k^7 + O(e_k^8), \end{aligned} \tag{22}$$

$$\begin{aligned} t =& \frac{w(z_k)}{w(\eta_k)} = (c_2^3 - c_2c_3)e_k^3 + (\alpha c_2 - 5c_2^4 + 9c_3c_2^2 - 2c_2^3 - 2c_2c_4)e_k^4 \\ &+ (15c_2^5 - 40c_2^3c_3 + 21c_2c_3^2 + \alpha(c_2^3 - 5c_2c_3) + 14c_2^2c_4 - 7c_3c_4 - 3c_2c_5)e_k^5 \\ &+ (-35c_2^6 + 125c_2^4c_3 - 110c_2^2c_3^2 + \cdots + \alpha(3c_2^4 - 9c_2^2c_3 + 7c_3^2 \cdots - 4c_2c_6))e_k^6 \\ &+ (72c_2^7 - 320c_2^5c_3 + \alpha(c_2^3 - 2c_2c_3) + 161c_2^4c_4 + \cdots + c_2(-126c_3^3 + \cdots - 5c_7))e_k^7 \\ &+ O(e_k^6), \end{aligned} \tag{23}$$

and

$$
\begin{aligned}
u =& \frac{w(z_k)}{w(y_k)} = (c_2^2 - c_3)e_k^2 - (2(c_2^3 - 2c_2c_3 + c_4) + \alpha c_2)e_k^3 \\
&+ (-2\alpha c_2^2 + c_2^4 - 3\alpha c_3 - 6c_2^2c_3 + c_2^2 + 5c_2c_4 - 3c_3)e_k^4 \\
&+ (4c_2^5 - 4c_2^3c_3 - 4c_2c_3^2 + \alpha(c_2^3 - 9c_2c_3 - 6c_4) - 4c_2^2 + \cdots + 4c_6)e_k^5 \\
&+ (-12c_2^6 + 38c_2^4c_3 - c_3^3 - \alpha(c_2^2 + c_3) - 13c_2^3c_4 + \cdots + c_2(3c_3c_4 + 7c_6 - 5c_7))e_k^6 \\
&+ O(e_k^7).
\end{aligned}
\tag{24}
$$

By the substitution of the expansions of v, t and u in (22)-(24) into the RVWF in (10)-(12) respectively, the following results were obtained.

$$
\begin{aligned}
&M(v) = M(0) + c_2M'(0)e_k + \left(2c_3M'(0) + \frac{1}{2}c_2^2\left(M''(0) - 6M'(0)\right)\right)e_k^2 \\
&+ \left(M'(0)(\alpha c_2 + 3c_4) + 2c_2c_3\left(M''(0) - 5M'(0)\right) + c_2^3\left(8M'(0) - 3M''(0) + \frac{M'''(0)}{6}\right)\right)e_k^3 \\
&+ \left(M'(0)(4c_5 - 8c_3^2) + \cdots + c_2^4\left(\frac{25}{2}M''(0) - \frac{3}{2}M'''(0) + M^{(iv)}(0) - 20M'(0)\right)\right)e_k^4 \\
&+ \left(M'(0)(5c_6 - 22c_3c_4) + \cdots + c_2^3c_3\left(83M^{(iv)}(0)\prime(0) - 118M'(0) - 11M'''(0) + 8\right)\right)e_k^5 \\
&+ O(e_k^6),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&P(t) = P(0) + \left(c_2^3 - c_2c_3\right)P'(0)e_k^3 - \left(5c_2^5 - \alpha c_2^2 - 9c_2^2c_3 + 2c_3^2 + 2c_2c_4\right)P'(0)e_k^4 \\
&+ \left(15c_2^5 - 40c_2^3c_3 + 21c_2c_3^2 + \alpha\left(c_2^3 - 5c_2c_3\right) + 14c_2^2c_4 - 7c_3c_4 - 7c_3c_4 - 3c_2c_5\right)P'(0)e_k^5 \\
&+ O(e_k^6),
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
&G(u) = G(0) + \left(c_2^2 - c_3\right)G'(0)e_k^2 - \left(-\alpha c_2 + 2\left(c_2^3 - 2c_2c_3 + c_4\right)\right)G'(0)e_k^3 \\
&+ \left(\left(-2\alpha c_2^2 + c_2^4 - 3\alpha c_3 - 6c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 3c_5\right)G'(0) + \frac{1}{2}\left(c_2^2 - c_3\right)G''(0)\right)e_k^4 \\
&+ \left(\left(4c_2^5 - 4c_2^3c_3 - 4c_2c_3^2 + \cdots - \left(c_2^2 - c_3\right)\left(\alpha c_2 + 2\left(c_2^3 - 2c_2c_3 + c_4\right)\right)\right)G''(0)\right)e_k^5 \\
&+ O(e_k^6).
\end{aligned}
\tag{27}
$$

The substitution of (16),(19), (20), (21) and (25)-(27) into the third step of (9) produces the following equations:

$$
\begin{aligned}
&s_{k+1} = z_k - \frac{w\left(z_k\right)}{2w\left[s_k, y_k\right] - w\left[s_k, \eta_k\right]}\left[M(v) \times P(t) \times G(u)\right] = \sigma - (c_2^3 - c_2c_3)\zeta e_k^4 \\
&+ (\alpha c_2^2\zeta + 2c_2^2\zeta + 2c_2c_4\zeta + c_2^4((4M(0) - M'(0))G(0)P(0) - 4) \\
&\qquad\qquad + c_2^2c_3(8 + G(0)(M'(0) - 8M(0)))P(0))e_k^5 \\
&+ (7c_3c_4\zeta + 2c_2^3c_4(6 + G(0)(M'(0) - 6M(0))P(0)) + \cdots + c_2^3(M'(0) - M(0))P(0))e_k^6 \\
&+ ((\alpha^2c_2^3\zeta + c_2^3c_4(4G'(0)M(0)P(0) + G(0)(34M(0) - 21M'(0) + M''(0))P(0)) - 40) \\
&- \cdots + c_3^2(-80 + 14G'(0)M(0)P(0) - G'(0)M'(0)P(0) \\
&\qquad + G(0)(62M(0)P(0) - 49M'(0)P(0) + 3M'(0)P(0) - M(0)P'(0))))e_k^7 \\
&+ (-6(-17c_4c_5\zeta - 13c_3c_6\zeta + \cdots \\
&\qquad + G(0)((90M'(0) - 27M''(0) + M'''(0))P(0) + 6M(0)(5P(0) + 2P'(0)))))e_k^8 \\
&+ O(e_k^9).
\end{aligned} \tag{28}
$$

where $\zeta = G(0)M(0)P(0) - 1$. For the IS (9) to converge with CO eight, the coefficients of e_k^j, $(4 \leq j \leq 7)$ in (28) must vanish. To achieve this, we need to solve for the solution of the set of equations below.

$$
\begin{aligned}
&\zeta = 0, \qquad M'(0) = 0, \qquad (G'(0) - G(0))M(0) = 0, \\
&4G(0)M(0) - 2G'(0)M(0) - G(0)M''(0) = 0, \qquad 2M(0) - M''(0) = 0, \\
&G(0)M(0)P'(0) - 2 = 0, \qquad (M(0))^2 - 1 = 0, \\
&M'''(0) + 12(M(0))^3 - 24M(0) = 0.
\end{aligned} \tag{29}
$$

The solutions that satisfy the set of equations in (29) are

$$
\begin{aligned}
&M(0) = P(0) = G(0) = 1, \quad M'(0) = 0, \quad M''(0) = 2, \quad M'''(0) = 12, \\
&M^{(iv)}(0) < \infty, \quad P'(0) = 2, \quad P''(0) = 2, \quad G'(0) = 1, \quad G''(0) < \infty.
\end{aligned} \tag{30}
$$

Substituting the solutions in (30) into (28), we then have

$$
\begin{aligned}
s_{k+1} =& \sigma + \frac{1}{2}(c_2^3\left(2ac_2^2 + 2c_2c_4 + 2c_2^2c_3\left(G''(0) - 6\right) - c_3^2\left(G''(0) - 2\right)\right. \\
&\left.-c_2^4\left(2M^{iv}(0) + G''(0) - 16\right)\right)e_k^8 + O\left(e_k^9\right) + O\left(e_k^9\right).
\end{aligned} \tag{31}
$$

From (31), the error equation of the IS (9) is obtained as

$$
\begin{aligned}
e_{k+1} =& \frac{1}{2}(c_2^3\left(2ac_2^2 + 2c_2c_4 + 2c_2^2c_3\left(G''(0) - 6\right) - c_3^2\left(G''(0) - 2\right)\right. \\
&\left.-c_2^4\left(2M^{iv}(0) + G''(0) - 16\right)\right)e_k^8 + O\left(e_k^9\right)
\end{aligned} \tag{32}
$$

Consequently, the equation in (32) implies that, IS (9) is of CO eight. This ends the proof. $\square$

Remark 3.1. For any three functions $M(v)$, $P(t)$ and $G(u)$ satisfying the conditions in Theorem 3.1, a CO eight can be obtained. Consider $M(v) = 1 + v^2 + 2v^3 + \varphi v^4$, $P(t) = 1 + 2t + \phi t^2$ and $G(u) = 1 + u + \tau u^2$, where $\varphi, \phi, \tau \in \Re$, will produce a parameterized family of an IS as:

$$
\begin{aligned}
y_k &= s_k - \frac{w(s_k)}{w[s_k,\eta_k]};\\
z_k &= y_k - \frac{w(y_k)}{2w[s_k,y_k] - w[s_k,\eta_k]};\\
s_{k+1} &= z_k - \frac{w(z_k)}{2w[s_k,y_k] - w[s_k,\eta_k]} \times \left(1 + \left(\frac{w(y_k)}{w(s_k)}\right)^2 + 2\left(\frac{w(y_k)}{w(s_k)}\right)^3 + \varphi\left(\frac{w(y_k)}{w(s_k)}\right)^4\right)\\
&\quad \times \left(1 + 2\left(\frac{w(z_k)}{w(\eta_k)}\right) + \phi\left(\frac{w(z_k)}{w(\eta_k)}\right)^2\right)\left(1 + \left(\frac{w(z_k)}{w(y_k)}\right) + \tau\left(\frac{w(z_k)}{w(y_k)}\right)^2\right).
\end{aligned} \tag{33}
$$

We present below some concrete members of IS (33) denoted as OU8a and OU8b with their respective error equations:

$$
\begin{aligned}
&OS8a: \qquad \varphi = \phi = \tau = 0\\
&\qquad e_{k+1} = \left(c_2^4 - c_2^2c_3\right)\left(ac_2^2 - 8c_2c_2^3 - 6c_2^2c_3 + c_3^2 + c_2c_4\right)e_k^8 + O\left(e_k^9\right)
\end{aligned}
$$

$$
\begin{aligned}
&OS8b: \qquad \varphi = \frac{1}{25}, \qquad \phi = 0, \qquad \tau = \frac{1}{30}\\
&\qquad e_{k+1} = \frac{\left(c_2^4 - c_2^2c_3\right)\left(300ac_2^2 + 2383c_2^4 - 1790c_2^2c_3 + 295c_3^2 + 300c_2c_4\right)e_k^8}{300} + O\left(e_k^9\right)
\end{aligned}
$$

The applicability of OS8a and OS8b are presented in Section 3.

Theorem 3.2. *Consider the scalar function $w : D \subset \Re \to \Re$ that is differentiable in D and has a simple solution $\sigma \in D$. If s_0 is close to σ, then the sequence $\{s_k\}_{k\geq 0}, (s_k \in D)$ of approximations generated by the IS in (13) will converge to σ with CO eight when $M(v)$ and $P(t)$ are jointly subjected to the conditions $M(0) = P(0) = 1, M'(0) = 0, M''(0) = 2$, $M'''(0) = 12$, $M^{(iv)}(0) < \infty$, and $P'(0) = 2$ ..*

Proof. From (17), (18), (20) and (21), we have

$$
\begin{aligned}
w[y_k, z_k] &= \frac{w(z_k) - w(y_k)}{z_k - y_k} = 1 + c_2^2e_k^2 + (2c_2c_3 - 2c_2^3)e_k^3\\
&+ c_2(\alpha c_2 + c_2c_3 - c_2 - c_2c_3 + 3c_4)e_k^4\\
&- c_2(\alpha(c_2^2 - 3c_3) + 4(3c_2^4 - 6c_2^2c_3 + c_3^2 + 3c_2c_4 - c_5))e_k^5\\
&+ (26c_2^6 - 69c_2^4c_3 + \cdots + 4c_2^2(7c_3^2 - 4c_5) + c_2(5c_6 - 18c_3c_4))e_k^6 + O(e_k^7).
\end{aligned} \tag{34}
$$

The substitution of the expansions in (21), (25), (27) and (34) into (13) yields

$$\begin{aligned} s_{k+1} &= \sigma - c_2(c_2^2 - c_3)\vartheta e_k^4 \\ &+ (\alpha\vartheta c_2^2 + 2\vartheta c_3^2 + 2\vartheta c_2 c_4 + c_2^4(4M(0)P(0) - M'(0)P(0) - 4) \\ &\qquad + c_2^2 c_3(8 - 8M(0)P(0) + M'(0)P(0)))e_k^5 \\ &+ (7\vartheta c_3 c_4 + \cdots + c_2(3\vartheta c_5 + c_3^2(18 - 18M(0)P(0) + 4M'(0))))e_k^6 \\ &+ (\alpha^2\vartheta c_2^2 + c_2^3 c_4(M''(0)P(0) - 21M'(0)P(0) + 38M(0)P(0) - 40) \\ &\quad \cdots + \cdots + c_3^2(76M(0)P(0) - 50M'(0)P(0) + 3M''(0)P(0) - M(0)P'(0) - 80))e_k^7 \\ &+ \frac{1}{6}(50c_3^2 c_4 - 17c_4 c_5 + \cdots + 2c_3^2(624M(0)P(0) - \cdots + \cdots + 3M'(0)P'(0) - 756))e_k^8 \\ &+ O(e_k^9), \end{aligned} \tag{35}$$

where $\vartheta = M(0)P(0) - 1$. For the terms involving e_k^j, $j = 4, 5, 6, 7$ in (35) to vanish, the solution of the following set of equations must hold.

$$\begin{aligned} &M(0)P(0) = 1, \quad M(0) \neq 0, \quad M(0)P(0) - 1 = 0 \quad M'(0) = 0 \\ &2M(0) - M''(0), \quad M(0)P'(0) - 2 = 0, \quad 12M(0) - M'''(0) = 0. \end{aligned} \tag{36}$$

The set of equations in (36) is satisfied when

$$\begin{aligned} &M(0) = P(0) = 1, \quad M'(0) = 0, \quad M''(0) = 2, \\ &M'''(0) = 12, \quad M^{(iv)}(0) < \infty, \quad P'(0) = 2. \end{aligned} \tag{37}$$

When the solution in (37) is substituted in (35), we have

$$s_{k+1} = \sigma + \left(c_2^4 - c_2^2 c_3\right)\left(\alpha c_2 - 4c_2 c_3 + c_4 - c_2^3\left(M^{(iv)}(0) - 7\right)\right)e_k^8 + O\left(e_k^9\right). \tag{38}$$

From (38), the error equation of the IS (13) becomes

$$e_{k+1} = \left(c_2^4 - c_2^2 c_3\right)\left(\alpha c_2 - 4c_2 c_3 + c_4 - c_2^3\left(M^{(iv)}(0) - 7\right)\right)e_k^8 + O\left(e_k^9\right). \tag{39}$$

Thus, from (39), the CO of IS presented in (13) is eight. This concludes the proof. □

Remark 3.2. For any two functions $M(v)$ and $P(t)$ satisfying the conditions in Theorem 3.2, a CO eight will be obtained. Consider $M(v) = 1 + v^2 + 2v^3 + \varphi v^4$ and

$P(t) = 1 + 2t + \phi t^2$, will produce another type of parameterised family of an IS as:

$$\begin{aligned} y_k &= s_k - \frac{w(s_k)}{w[s_k, \eta_k]}; \\ z_k &= y_k - \frac{w(y_k)}{2w[s_k, y_k] - w[s_k, \eta_k]}; \\ s_{k+1} &= z_k - \frac{w(z_k)}{w[z_k, y_k]} \left(1 + \left(\frac{w(y_k)}{w(x_k)}\right)^2 + 2\left(\frac{w(y_k)}{w(x_k)}\right)^3 + \varphi \left(\frac{w(y_k)}{w(x_{x_k})}\right)^4\right) \\ &\quad \times \left(1 + 2\left(\frac{w(z_k)}{w(\eta_k)}\right) + \phi \left(\frac{w(z_k)}{w(\eta_k)}\right)^2\right) \end{aligned} \tag{40}$$

Two concrete members of IS (40) which are denoted as OU8c, and OU8d with their respective error equations are:

$OS8c: \quad \varphi = \phi = 0,$

$$e_{k+1} = \left(c_2^4 - c_2^2 c_3\right)\left(ac_2 + 7c_2c_2^3 - 4c_2c_3 + c_4\right)e_k^8 + O\left(e_k^9\right)$$

$OS8d: \quad \varphi = 7, \quad \phi = 0$

$$e_{k+1} = \left(c_2^4 - c_2^2 c_3\right)\left(ac_2 - 4c_2c_3 + c_4\right)e_k^8 + O\left(e_k^9\right)$$

4. The IS implementation

This section provides the implementation and applicability of the developed IS (OS8a, OS8b, OS8c and OS8d) with $\alpha = 0.001$. Some NL equations recently used in testing developed IS in the literature [3, 4, 5] were also used to carry out the implementation of the developed IS. THe computational performance of the developed IS were compared with that of some good existing IS of same CO. Three measures used for comparison includes: Number of iteration required by an IS to attain convergence (IT), absolute value of function of k iteration point ($|w(s_k)|$) and computational CO (ν_{coc}) put forward by Jay [16] given as

$$\nu_{coc} \approx \frac{\ln |w(s_{k+1})/w(s_k)|}{\ln |w(s_k)/w(s_{k-1})|}. \tag{41}$$

The IS used for comparison includes the one in Soleymani [10] (SL8) CO eight IS:

$$\begin{aligned} y_k &= s_k - \frac{w(s_k)}{w[s_k, \Delta]}, \qquad \Delta = s_n + w(s_k), \\ z_n &= y_k - \frac{w(y_k)}{w[s_k, \Delta]}\left[1 + \frac{2 + w[s_k, \Delta]}{1 + w[s_k, \Delta]}\frac{w(y_k)}{w(s_k)}\right], \\ s_{k+1} &= z_n - \frac{w(z_k)}{w[s_k, z_k]}\left[1 + \frac{1}{1 + w[s_k, \Delta]}\left(\frac{w(y_k)}{w(s_k)}\right)\right], \end{aligned} \tag{42}$$

Yasmin et al, [13] (YZS8) (Eq. 2.12) CO eight IS:

$$\begin{aligned} y_k &= s_k - \frac{w(s_k)}{w\left[\eta_k, s_k\right]}, \\ z_k &= y_k - \frac{w(y_k)}{2f\left[y_k, s_k\right] - w\left[\eta_k, s_k\right]}, \\ s_{k+1} &= z_k - \frac{w(z_k)}{\Lambda_k}, \end{aligned} \tag{43}$$

where

$\Lambda_k = w\left[z_k, s_k\right]\left(2 + \frac{z_k - s_k}{z_k - y_k}\right) - \frac{(z_k - s_k)^2}{(y_k - s_k)(z_k - y_k)} w\left[y_k, s_k\right] + f\left[\eta_k, s_k\right]\left(\frac{z_k - y_k}{y_k - s_k}\right)$,

and Soleymani and Shateyi (SS8) [11] (Eq. 2.8) CO eight IS:

$$\begin{aligned} y_k &= s_k - \frac{w(s_k)}{w\left[\Psi_k, s_k\right]}, \qquad \Psi_k = s_k + \alpha w(s_k), \\ z_k &= y_k - \frac{w(y_k)w(\Psi)}{\left(w(\Psi_k) - w(y_k)\right) w\left[\Psi_k, s_k\right]}, \\ s_{k+1} &= z_k - \frac{w(z_k)w((\Psi_k)}{\left(w(\Psi_k) - w(y_k)\right) w\left[\Psi_k, s_k\right]}\Theta_k, \end{aligned} \tag{44}$$

where

$\Theta_k = \left\{\left(1 + \frac{w(z_k)}{w(y_k)}\right)\left(1 + \frac{w(z_k)}{w(\Psi_k)}\right)\left(1 + \frac{w(z_k)}{w(s_k)}\right)\Lambda\right\}, \Lambda = \left(1 + \left(1 + \alpha w\left[s_k, \Psi_k\right]\right)\left(\frac{w(y_k)}{w(\Psi_k)}\right)^2\right)$.

In order to obtain better approximation of NL equation solution and to reduce loss of significant figures (s.f), 2000 s.f was used in the execution of developed program code in Maple 2017 software environment, where $|w(s_k)| < 10^{-1000}$ was used as stopping criterion.

The NL equations utilized for the IS implementation includes:

Example 4.1. $w_1(s) = -2 + (s-1)^3 = 0, \quad \sigma = 2.2599\cdots$, see [4].

Example 4.2. $w_2(s) = 1 - s^2 + sin^2(s) = 0, \quad \sigma = 1.4044\cdots$, see [4].

Example 4.3. $w_3(s) = 1 - 2s - ln(s) - 7 = 0, \quad \sigma = 4.2199\cdots$, see [5].

Example 4.4. $w_4(s) = -.75e^{-0.05s} + 1 = 0, \quad \sigma = -5.753\cdots$, see [5].

Example 4.5. $w_5(s) = 5 - 5e^{-s} - s = 0, \quad \sigma = 4.9651\cdots$, see [3].

The computation results obtained when the IS were used to solve Example 4.1-4.5 are presented in Table 1-4. Observe that all the developed IS solved the NL equations in Example 4.1-4.5 just as the compared IS. Furthermore,the calculated CO obtained using (41) on the computation results for the developed IS (see the last column of Table 1-4) agrees with the theoretical order of convergence obtained in Section 3.

Table 1. Comparison of IS results for Examples 4.1-4.4

IS	s_0	$\|w(s_1)\|$	$\|w(s_2)\|$	$\|w(s_3)\|$	$\|w(s_4)\|$	$\|w(s_5)\|$	ν_{coc}
SL8		$9.4e-1$	$6.1e-03$	$1.2e-016$	$1.8e-0112$	$2.2e-783$	8.0
YZS8		$1.0e-2$	$2.9e-22$	$9.6e-179$	$1.4e-1430$	-	8.0
SS8		$1.8e-2$	$3.0e-19$	$1.7e-153$	$2.0e-1227$	-	8.0
OS8a	3.0	$7.5e-3$	$1.6e-22$	$5.7e-180$	$1.6e-1439$	-	8.0
OS8b		$7.5e-3$	$1.5e-22$	$4.5e-180$	$2.4e-1440$	-	8.0
OS8c		$7.4e-3$	$1.3e-22$	$1.3e-180$	$1.4e-1444$	-	8.0
OS8d		$1.0e-3$	$3.1e-21$	$2.9e-169$	$1.7e-1353$	-	8.0
SL8		$5.6e-2$	$3.2e-12$	$7.3e-084$	$2.3e-0585$	-	8.0
YZS8		$1.8e-4$	$1.6e-34$	$7.4e-275$	-	-	8.0
SS8		$2.2e-3$	$1.5e-24$	$1.0e-193$	$3.3e-1547$	-	8.0
OS8a	2.0	$1.4e-3$	$2.4e-26$	$2.3e-208$	$1.5e-1664$	-	8.0
OS8b		$1.4e-3$	$2.3e-26$	$1.2e-208$	$7.7e-1667$	-	8.0
OS8c		$1.0e-3$	$1.7e-27$	$1.4e-217$	$2.8e-1738$	-	8.0
OS8d		$1.2e-3$	$1.9e-26$	$4.9e-209$	$1.1e-1669$	-	8.0
SL8		$2.1e-11$	$1.6e-083$	$1.8e-0588$	-	-	8.0
YZS8		$5.5e-15$	$6.4e-126$	$2.1e-1013$	-	-	8.0
SS8		$1.3e-14$	$1.1e-122$	$4.3e-987$	-	-	8.0
OS8a	4.0	$1.1e-14$	$5.1e-123$	$7.7e-990$	-	-	8.0
OS8b		$1.1e-14$	$4.5e-123$	$2.8e-990$	-	-	8.0
OS8c		$6.9e-15$	$4.9e-175$	$3.3e-1006$	-	-	8.0
OS8d		$7.2e-15$	$9.3e-125$	$8.2e-1005$	-	-	8.0
SL8		$3.3e-4$	$10.0e-26$	$2.1e-176$	$3.5e-1231$	-	8.0
YZS8		$3.5e-7$	$3.5e-55$	$4.1e-439$	-	-	8.0
SS8		$5.8e-6$	$3.4e-44$	$4.5e-350$	-	-	8.0
OS8a	1.0	$5.0e-6$	$4.6e-45$	$2.4e-357$	-	-	8.0
OS8b		$4.9e-6$	$4.1e-45$	$9.2e-358$	-	-	8.0
OS8c		$4.9e-6$	$4.1e-45$	$1.0e-357$	-	-	8.0
OS8d		$1.6e-5$	$1.7e-40$	$1.1e-320$	-	-	8.0

4.1. Real life problems applications. *Application 1:* (Chemical equilibrium [17]) Consider the equation of obtaining the fraction (fractional conversion) of the nitrogen-hydrogen feed that gets converted to ammonia presented in [17]. Suppose we have the pressure of 250 atm and temperature of 500^0C, then the problem involves obtaining the solution of the equation

$$w(s) = \frac{8s^2(4-s)^2}{(2-s)(6-3s)^2} - 0.186 = 0. \tag{45}$$

TABLE 2. Comparison of IS results for Examples 4.5

IS	s_0	$\|w(s_1)\|$	$\|w(s_2)\|$	$\|w(s_3)\|$	$\|w(s_4)\|$	ν_{coc}
SL8		$1.1e-1$	$3.8e-11$	$2.2e-077$	$5.3e-541$	8.0
YZS8		$9.4e-5$	$5.2e-42$	$5.0e-340$	-	8.0
SS8		$8.2e-5$	$3.2e-41$	$1.8e-332$	-	8.0
OS8a	0.5	$2.2e-6$	$2.5e-54$	$7.5e-438$	-	8.0
OS8b		$3.3e-6$	$7.4e-53$	$4.3e-426$	-	8.0
OS8c		$2.0e-5$	$2.1e-45$	$1.9e-366$	-	8.0
OS8d		$2.6e-5$	$2.9e-45$	$8.5e-365$	-	8.0

The above equation can be reduced to a polynomial equation as:

$$w(s) = s^4 - 7.79075s^3 + 14.744s^2 + 2.511s - 1.674 = 0. \tag{46}$$

The equation in (46) has four solutions. These are : $s_1 = 0.27776\ldots$; $s_2 = -0.384094\ldots$; $s_3 = 3.94854 \pm 316124i$ and $s_4 = 3.94854 \pm 0.316124i$. But the factional conversion value must lie between 0 and 1. Therefore, the first solution $s_1 = 0.27776\ldots$ satisfies this condition because it is meaningful in practice. The computation results by the various IS when applied to solve the problem in (46) are presented in Table 3.

TABLE 3. Comparison of IS results for Application 1

IS	s_0	$\|w(s_1)\|$	$\|w(s_2)\|$	$\|w(s_3)\|$	$\|w(s_4)\|$	$\|w(s_5)\|$	ν_{coc}
SL8		$3.8e-1$	$2.5e-05$	$4.8e-033$	$3.8e-0227$	$7.9e-1586$	8.0
YZS8		$3.6e-4$	$2.9e-34$	$3.9e-275$	-	-	8.0
SS8		$2.9e-3$	$3.4e-26$	$1.2e-209$	$3.6e-1677$	-	8.0
OS8a	0.8	$3.0e-3$	$2.9e-26$	$2.2e-210$	$2.4e-1683$	-	8.0
OS8b		$3.0e-3$	$2.5e-26$	$7.2e-211$	$3.3e-1687$	-	8.0
OS8c		$1.3e-3$	$2.8e-29$	$1.4e-234$	$5.0e-1877$	-	8.0
OS8d		$1.3e-3$	$4.9e-29$	$1.7e-232$	$3.8e-1860$	-	8.0

Application 2:(Conversion in a chemical reactor) Consider the example on chemical reaction formulation presented in [18] as

$$w(s) = 4.45977 - 5ln\left(\frac{0.4\,(1-s)}{0.4-0.5s}\right) + \frac{s}{1-s} = 0, \tag{47}$$

where s (bounded between 0 and 1) is the species fractional conversion in a chemical reactor. A solution satisfying (47) is $s = 0.7573962463\ldots$. The the computational results obtained for the IS are given in Table 4.

TABLE 4. Comparison of IS results for Application 2

IS	s_0	$\lvert w(s_1)\rvert$	$\lvert w(s_2)\rvert$	$\lvert w(s_3)\rvert$	$\lvert w(s_4)\rvert$	ν_{coc}
SL8				Diverged		-
YZS8		$7.0e-5$	$3.5e-40$	$1.2e-322$	$1.3e-1998$	8.0
SS8		$1.2e-3$	$3.1e-29$	$6.1e-234$	$1.4e-1871$	8.0
OS8a	0.78	$1.2e-3$	$5.1e-30$	$6.1e-241$	$2.4e-1928$	8.0
OS8b		$1.2e-3$	$6.6e-30$	$4.9e-240$	$4.2e-1921$	8.0
OS8c		$3.3e-3$	$1.6e-26$	$6.3e-213$	$3.9e-1704$	8.0
OS8d		$3.3e-3$	$7.5e-26$	$5.9e-207$	$8.2e-1656$	8.0

5. Conclusion

This work put forward two families of optimal order eight IS that require no evaluation of functions derivatives. The techniques used in their development includes the approximation of function derivatives with interpolating polynomial, divided difference and weight functions. The flexibility of the weight functions, which involves parameters, enables the construction of plethora of concrete forms of the developed IS. The computational performance of the developed IS as compared with some well established existing IS of same CO, shows that they can be used as good alternatives.

References

[1] W. H. Chanu, S. Panday and G. Thangkhenpau, *Development of optimal iterative methods with their applications and Basins of attraction*, Symmetry, **14**(10) (2022), 1-21.

[2] Y. A. Noori, H. A. Ali and C. Park, *A new fifth order iterative method free from second derivative for solving nonlinear equations*, J. Appl. Math. Comput., **68**(93) (2021), 2877-2886.

[3] O. Ogbereyivwe and V. Ojo-Orobosa, *Family of optimal two-step fourth order iterative method and its extension for solving nonlinear equations*, J. Interdiscip. Math. **24** (2021), 1347-1365.

[4] O. Ogbereyivwe and O. Izevbizua, *A three-parameter class of power series based iterative method for approximation of nonlinear equations solution*, Iran. J. Numer. Anal. Optim., Article in press, doi:10.22067/IJNAO.2022.74901.1095.

[5] O. Ogbereyivwe and S. A. Ogumeyo, *Extension of the double Newton's method convergence order via the bi-variate power series weight function for solving nonlinear models*, Math. Anal. Contemp. Appl., **4**(3) (2022), 35–47.

[6] O. Ogbereyivwe and V. Ojo-Orobosa, *Families of means-based modified Newtons method for solving nonlinear models*, Punjab Univer. J. Math. **53**(11) (2021), 779-791.

[7] J. F. Traub, *Iterative methods for the solution of equations*, Chelsea Publishing Company, Newyork, 1977.

[8] U. F. Steffensen, *Remarks on iteration*, Skand. Aktuarrietiddskr, **16** (1933), 64-72.

[9] B. Panday and J. P. Jaiswal, *New seventh and eight order derivative free methods for solving nonlinear equations*, Tbilisi Math. J., **10**(4) (2017), 103-115.

[10] F. Soleymani and S. Shateyi, *Two classes of iterative schemes for approximating simple roots*, J. Appl. Sci. **11**(9) (2010), 3442-3446.

[11] F. Soleymani and S. Shateyi, *Two optimal eighth-order derivative-free classes of iterative methods*, Abstr. Appl. Anal., **2012** (2012), 1-14.
[12] G. Thangkhenpau and S. Panday, *Optimal eight order derivative-free family of iterative methods for solving nonlinear equations*, IAENG Int. J. Comput. Sc., **50**(1) (2023), 1-37.
[13] N. Yasmin, F. Zafar and S. Akram, *Optimal derivative-free root finding methods based on the Hermite interpolation*, J. Nonlinear Sci. Appl., **9** (2016), 4427-4435.
[14] P. Sivakumar, K. Madhu and J. Jayaraman, *Optimal eight and sixteenth order iterative methods for solving nonlinear equation with basins of attraction*, Appl. Math. E-Notes **21** (2021), 320-343.
[15] H. T. Kung and J. F. Traub, *Optimal order of one-point and multipoint iteration*, J. Assoc. Comput. Machin. **21** (1974), 643-651.
[16] L. O. Jay, *A note on Q-order of convergence*, BIT Numer. Math., **41** (2001), 422-429.
[17] G. V. Balaji and J. D. Seader, *Application of interval Newton's method to chemical engineering problems*, Reliable Comput., **1**(3) (1995), 215–223.
[18] M. Shacham and E. Kehat, *An iteration method with memory for the solution of a non-linear equation*, Chem. Engin. Sci., **27**(11) (1972), 2099–2101.

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