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On algebraic bounds for exponential function with applications

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Abstract. In this paper, we establish algebraic bounds of the ratio-type in nature for the natural exponential function e^x involving two parameters, a and n, which become optimal as $a \to 0$ or $n \to \infty$. The proof is mainly based on Chebyshev's integral inequality and properties of the incomplete gamma function. Subsequently, we focus on the simple case obtained with $n = 1$, with comparisons to existing literature results. For the applications, we provide alternative proofs of inequalities involving ratio functions of trigonometric and hyperbolic functions. Graphics are given to illustrate the theory.

1. Introduction

The exponential function is "the most significant function in mathematics", according to the prestigious mathematician Walter Rudin, due to its frequent occurrence in both pure and practical mathematics (see [11]). It can be found in a wide range of applications in the fields of physics, chemistry, computer science, engineering, biology, medicine, finance, and economics. From a mathematical viewpoint, when taken as such, it is a simple function. However, when it appears in a sophisticated mathematical expression (integral, series, partial derivative equations, etc.), it can be particularly hard to manage. For this reason, numerous efforts have been made to find sharp bounds of different natures. The topic is vast; numerous bounds for the exponential function have already been established. See, e.g.,

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[1, 2, 3, 5, 6, 7, 8, 9]. If we focus on ratio-type bounds, the inequality

$$e^x \leq \frac{1}{1-x}, \quad x \in (0,1), \tag{1}$$

is the most famous and simplest one. Recent advances on this topic contain the result shown by S.-H. Kim [7]. It can be stated as follows: for $a \in (1/2, 1)$, we have

$$e^x \leq U(a,x) \leq \frac{1}{1-x}, \quad x \in (0,1), \tag{2}$$

where

$$U(a,x) = (1-a) + a\left(\frac{1+(1-a)x}{1-ax}\right)^{1/a}.$$

This result reveals that there is again room for improvement for the inequality in (1). The present paper aims to fill this gap by demonstrating new and sharp ratio lower and upper bounds for e^x, upper bounds that also improve $(1-x)^{-1}$. These results can be used to derive sharp bounds for other functions of interest. This claim will be illustrated by the consideration of ratio functions involving trigonometric and hyperbolic functions. In addition to the detailed proofs, graphics are provided to support the theory when adapted.

The organization of the paper is as follows: Section 2 presents the main result of the paper. A special case is emphasized in Section 3. Some bound comparisons are made in Section 4. Applications beyond the exponential function are given in Section 5. A conclusion is formulated in Section 6.

2. Results

Inspired by the result of S.-H. Kim [7], but with a completely different approach in terms of proof, we establish the following proposition:

Proposition 2.1. *Let $a > 0$ and n be a positive integer. Let us set*

$$V(a,n,x) := \left(\frac{(-1)^{n+1}(n+1)! + a^n x^n}{a^n x^n + (-1)^{n+1}(n+1)!\sum_{k=0}^{n}(-1)^k a^k x^k/(k!)}\right)^{1/a}, \tag{3}$$

provided that it exists. Then,

- *for $x > 0$ and n odd such that*

$$x^n > \frac{1}{a^n}(n+1)!\sum_{k=0}^{n}(-1)^{k+1}a^k\frac{x^k}{k!},$$

we have

$$e^x \leq V(a,n,x), \tag{4}$$

- *for $x > 0$ and n even such that*

$$x^n < \frac{1}{a^n}(n+1)!,$$

we have

$$e^x \geq V(a,n,x). \tag{5}$$

The upper bound $V(a,n,x)$ is optimal as $a \to 0$, or $n \to \infty$.

Proof. The proof is based on the famous Chebyshev's integral inequality [9] and some properties of the incomplete gamma function. To begin, we recall the mentioned Chebyshev's integral inequality. If f, g are two integrable functions defined on $[p,q]$ with $p, q > 0$ such that both f, g are either increasing or decreasing then

$$\int_p^q f(t)g(t)\,dt \geq \frac{1}{q-p}\int_p^q f(t)\,dt \int_p^q g(t)\,dt. \tag{6}$$

The inequality in (6) is reversed if one of the functions is decreasing and the other one is increasing. The equality holds if and only if one of the functions is constant.

In this proof, we put $p = 0$, $q = x$ and $f(t) = t^n$, $g(t) = e^{at}$ where $a > 0$, $x > 0$ in (6). So we have

$$\int_0^x t^n\, e^{at}\,dt \geq \frac{1}{x}\int_0^x t^n\,dt \int_0^x e^{at}\,dt.$$

It is worth noting that the first integral term is connected with the incomplete gamma function: $\gamma(m,x) = \int_0^x t^{m-1}e^{-t}dt$. By the change of variables $y = -at$, and a well-known decomposition of the incomplete gamma function taken at an integer, we obtain

$$\int_0^x t^n e^{at}dt = \frac{(-1)^{n+1}}{a^{n+1}}\int_0^{-ax} y^n e^{-y}dy = \frac{(-1)^{n+1}}{a^{n+1}}\gamma(n+1,-ax)$$
$$= \frac{(-1)^{n+1}}{a^{n+1}}n!\left(1 - e^{ax}\sum_{k=0}^{n}\frac{(-1)^k a^k x^k}{k!}\right).$$

On the hand, it is immediate that

$$\int_0^x t^n dt = \frac{x^{n+1}}{n+1}, \qquad \int_0^x e^{at}dt = \frac{1}{a}(e^{ax}-1).$$

So, by the Chebyshev's integral inequality, we have

$$\frac{(-1)^{n+1}}{a^{n+1}}n!\left(1 - e^{ax}\sum_{k=0}^{n}\frac{(-1)^k a^k x^k}{k!}\right) \geq \frac{1}{x}\times\frac{x^{n+1}}{n+1}\times\frac{1}{a}(e^{ax}-1),$$

which can be arranged as

$$(-1)^{n+1}(n+1)! + a^n x^n \geq e^{ax}\left(a^n x^n + (-1)^{n+1}(n+1)!\sum_{k=0}^{n}\frac{(-1)^k a^k x^k}{k!}\right). \tag{7}$$

On the one hand, for $x > 0$ and n odd such that

$$x^n > \frac{1}{a^n}(n+1)! \sum_{k=0}^{n} (-1)^{k+1} a^k \frac{x^k}{k!},$$

the left and right terms in (7) are strictly positive, and we have

$$e^x \leq V(a, n, x).$$

On the other hand, for $x > 0$ and n even such that

$$x^n < \frac{1}{a^n}(n+1)!,$$

the left and right terms in (7) are strictly negative (the negativity of the left term implying the one of the right term, and the exponential function is always positive), so the inequality reverse after division, and we get

$$e^x \geq V(a, n, x).$$

Also, we have

$$\lim_{a \to 0} V(a, n, x) = \lim_{n \to \infty} V(a, n, x) = e^x.$$

Thus $V(a, n, x)$ is an optimal upper bound for e^x. This ends the proof of Proposition 2.1. $\square$

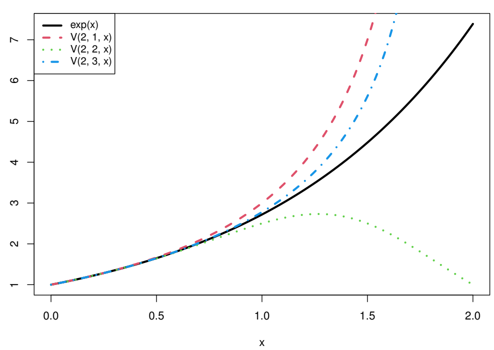

FIGURE 1. Graphs of e^x and the bounds $V(a, 1, x)$, $V(a, 2, x)$ and $V(a, 3, x)$ in (3) for $a = 1$ and $x \in (0, 2)$.

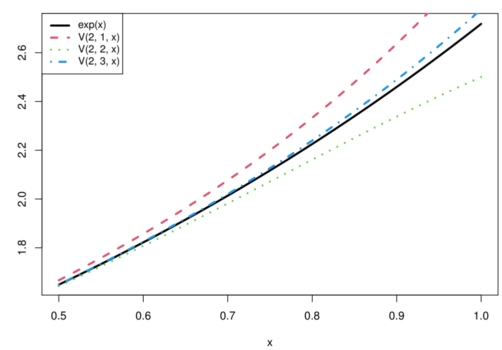

FIGURE 2. Graphs of e^x and the bounds $V(a, 1, x)$, $V(a, 2, x)$ and $V(a, 3, x)$ in (3) for $a = 1$ and $x \in (0.5, 1)$.

We now list the expression for $V(a, n, x)$ for the first values of n. We have

$$V(a, 1, x) = \left(\frac{2 + ax}{2 - ax} \right)^{1/a},$$

which is valid if $x \in (0, 2/a)$, and in this case, $e^x \leq V(a, 1, x)$.

We have

$$V(a,2,x)=\left(\frac{6-a^2x^2}{2a^2x^2-6ax+6}\right)^{1/a},$$

which is valid for $x\in(0,\sqrt{6}/a)$, and in this case, due to the value of $n=2$, we have $e^x\geq V(a,2,x)$. We have

$$V(a,3,x)=\left(\frac{a^3x^3+24}{3(2-ax)(a^2x^2-2ax+4)}\right)^{1/a},$$

which is valid for $x\in(0,2/a)$, and in this case, $e^x\leq V(a,3,x)$.

These findings are illustrated for $a=1$, with $x\in(0,2)$ in Figure 1 on the one hand, and a zoom work for $x\in(0.5,1)$ in Figure 2 on the other hand.

3. Discussion on the case $n=1$

The simplest case $n=1$ seems to have not received a lot of attention from the literature. It appeared in [**9**, p. 269] and also in the inequality sheets in [**10**]. After a bit of algebra, for $a=1$, it turns out to be equivalent to the following more well-known hyperbolic inequality: $\tanh x<x$ for any $x\in(0,1)$. In this section, we discuss this special case in the light of the recent findings of the literature.

The following proposition implies that the upper bound of e^x in (5) is sharper than that of (1).

Proposition 3.1. *Let $x\in(0,1)$ and $a\in(0,1)$. Then it holds that*

$$V(a,1,x):=\left(\frac{2+ax}{2-ax}\right)^{1/a}<\frac{1}{1-x}. \tag{8}$$

Proof. We need to prove that

$$\frac{2+ax}{2-ax}<\left(\frac{1}{1-x}\right)^a,$$

i.e.,

$$(2+ax)(1-x)^a<2-ax$$

or

$$ax+(2+ax)(1-x)^a-2<0$$

for a and x in $(0,1)$. Let $f(x)=ax+(2+ax)(1-x)^a-2$. By differentiation, we get

$$f'(x)=a\left(1-(2+ax)(1-x)^{a-1}+(1-x)^a\right).$$

Similarly, we have

$$f''(x)=(a-1)(2+ax)(1-x)^{a-2}-2a(1-x)^{a-1}<0,$$

as $x\in(0,1)$ and $a\in(0,1)$. From this, we conclude that $f'(x)$ is strictly decreasing in $(0,1)$. Hence $f'(x)<f'(0)=0$ for $x>0$. Consequently, $f(x)$ is strictly decreasing in $(0,1)$ and we write $f(x) < f(0) = 0$, for $x > 0$. This completes the proof of Proposition 3.1. □

4. Some comparison

Clearly, we have $V(a,1,x) \leq (1-x)^{-1} = U(1,x)$ by Proposition 3.1. Since $V(a,1,x)$ is optimal as $a \to 0$, we have

$$V(a,1,x) := \left(\frac{2+ax}{2-ax}\right)^{1/a} \leq \frac{2+x}{2-x}; \text{ for } a \in (0,1] \text{ and } x \in (0,2/a).$$

Now by A. M. - G. M. inequality we get

$$U(1/2,x) = \frac{1}{2} + \frac{1}{2}\left(\frac{2+x}{2-x}\right)^2 \geq \frac{2+x}{2-x} \geq V(a,1,x)$$

for $a \in (0,1)$ and $x \in (0,2/a)$.

It should be noted that our upper bounds are valid in a larger interval if $a \in (0,1)$. Figures 3, 4, 5, 6 show that our upper bound $V(a,1,x)$ for e^x is sharper than the corresponding one $U(a,x)$.

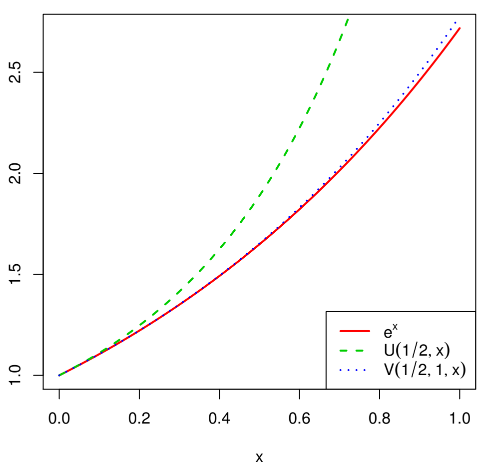

Figure 3. Graphs of upper bounds of e^x in (2) and (8) for $a = 1/2$ and $x \in (0,1)$.

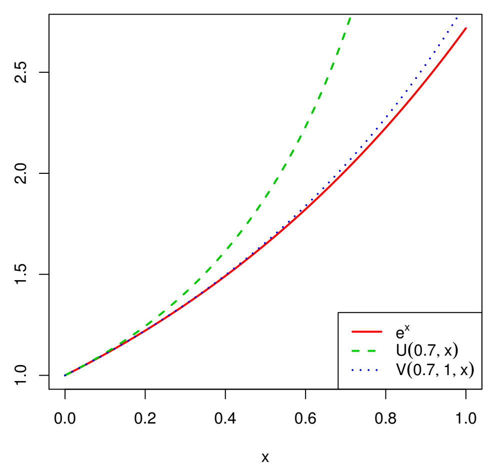

Figure 4. Graphs of upper bounds of e^x in (2) and (8) for $a = 0.7$ and $x \in (0,1)$.

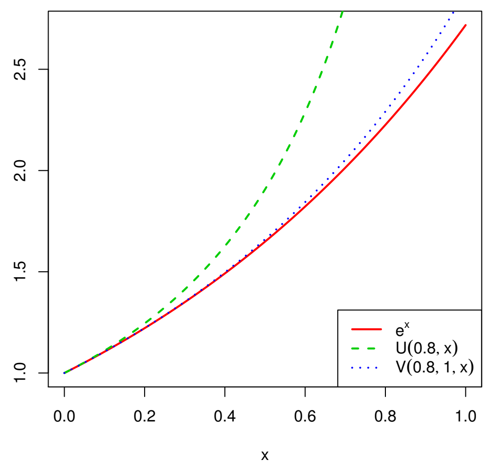

FIGURE 5. Graphs of upper bounds of e^x in (2) and (8) for $a = 0.8$ and $x \in (0, 1)$.

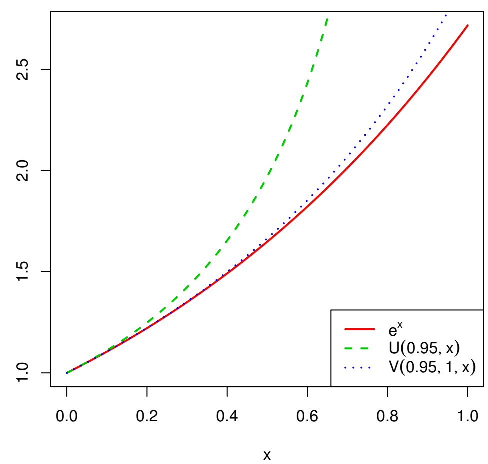

FIGURE 6. Graphs of upper bounds of e^x in (2) and (8) for $a = 0.95$ and $x \in (0, 1)$.

5. Applications

In this section, we give applications of Proposition 2.1. The following inequalities:

$$e^{x^2} \leq \frac{\cosh x}{\cos x}, \quad x \in \left(0, \frac{\pi}{2}\right) \tag{9}$$

and

$$e^{x^2/3} \leq \frac{\sinh x}{\sin x}, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{10}$$

were recently established by Bagul at. al. [4]. The proofs of these inequalities are based on monotonicity of appropriately chosen functions. Here we give alternative proofs of (9) and (10). First, we write the inequality in (5) as follows:

$$e^{ax} \leq \frac{1 + (a/2)x}{1 - (a/2)x}, \quad x \in \left(0, \frac{2}{a}\right). \tag{11}$$

Using the following infinite products:

$$\cosh x = \prod_{k=1}^{\infty}\left(1 + \frac{4}{\pi^2(2k-1)^2}x^2\right) \text{ and } \cos x = \prod_{k=1}^{\infty}\left(1 - \frac{4}{\pi^2(2k-1)^2}x^2\right)$$

and the inequality in (11), we have

$$\frac{\cosh x}{\cos x} = \prod_{k=1}^{\infty} \frac{\left(1 + \frac{4}{\pi^2(2k-1)^2}x^2\right)}{\left(1 - \frac{4}{\pi^2(2k-1)^2}x^2\right)} \geq \prod_{k=1}^{\infty} e^{\frac{8}{\pi^2(2k-1)^2}x^2}$$
$$= e^{\sum_{k=1}^{\infty} \frac{8x^2}{\pi^2} \frac{1}{(2k-1)^2}} = e^{\frac{8x^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}} = e^{x^2}$$

since $\sum_{k=1}^{\infty}(2k-1)^{-2} = \pi^2/8$. This yields the inequality in (9).

Similarly, using the following infinite products:

$$\frac{\sinh x}{x} = \prod_{k=1}^{\infty}\left(1 + \frac{1}{\pi^2 k^2}x^2\right) \text{ and } \frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1 - \frac{1}{\pi^2 k^2}x^2\right)$$

with the inequality in (11) and $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$, we get

$$\frac{\sinh x}{\sin x} = \prod_{k=1}^{\infty} \frac{(1 + \frac{1}{\pi^2 k^2}x^2)}{(1 - \frac{1}{\pi^2 k^2}x^2)} \geq \prod_{k=1}^{\infty} e^{\frac{2}{\pi^2 k^2}x^2}$$
$$= e^{\sum_{k=1}^{\infty} \frac{2x^2}{\pi^2} \frac{1}{k^2}} = e^{\frac{2x^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}} = e^{x^2/3}.$$

This gives the desired inequality in (10).

6. Conclusion

New sharp bounds for the exponential functions are rare because a lot already exist. In this paper, we nevertheless contribute to the topic by establishing new and sharp lower and upper bounds of the ratio-type. The advantages of these traits are being original, flexible, and sharp. "Original" because of the proof scheme; Chebyshev's integral inequality and incomplete gamma function results are thoroughly combined to obtain them, "flexible" in the sense that they depend on two tuning parameters, and "sharp" in the sense that they improve some comparable bounds of the literature. As illustrated in our application, these bounds can be used to evaluate completely different functions, such as the ratio of trigonometric and hyperbolic functions. One can also think of special integral functions mixing ratio and exponential functions, with the ratio bounds being more appropriate in this case. These ideas about perspectives need further development, which we will leave for future work.

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