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Solving conformable fractional Sturm-Liouville equations using one class of special polynomials and special functions

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Abstract. The objective of this paper is to solve conformable fractional Sturm-Liouville equations using one class of special polynomials and special functions introduced in [**13**]. Also, the connection between Mittag-Leffler functions and special polynomials are established and conformable fractional derivatives of certain Mittag-Leffler functions are determined.

1. Introduction

The fractional differential calculus has been developed to describe different physical phenomena. In the last three decades, the fractional differential calculus has became of great importance in many fields of science and engineering, for example mechanics, electricity, chemistry, biology, economics, control theory and signal and image processing [**14**], [**15**]. The fractional differential calculus is nowadays one of the most intensively developing areas of mathematical analysis, including several definitions of fractional operators like Riemann-Liouville, Caputo, and Grünwald-Letnikov [**5**]. These fractional derivatives are complicated, especially Grünwald-Letnikov where some of the basic properties that usual derivatives have such as the product rule and the chain rule are lost. This was one of the motivations for many authors to introduce a new definition of the fractional derivative that would preserve these properties.

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In [11] authors introduced a definition of a conformable fractional derivative (CFD) which is a natural extension of the usual derivative and for which hold multiplication rule, division rule, fractional Rolle theorem, and fractional average value theorem. This definition coincides with the known fractional derivatives of polynomials. Later, in [1] are defined the right and left conformable fractional derivatives,the fractional chain rule, and fractional integrals of higher orders. In [2] the authors discussed the Sturm-Liouville problems in the frame of conformable derivatives (see also [9]).

In this paper, we focus our attention on the conformable fractional Sturm-Liouville equations ([12], [16])

$$(t^2+1)^2 D_t^{2a} y(t) + \left(2t^{2-a}(1-n) - 2nt - (1-a)t^{-a}(1+t^2)\right) D_t^a(y(t)) \\ + 2nt^{2-2a}(2-a)(2n-1)y(t) = 0, \quad t>0, \tag{1}$$

and

$$(t^2+1)^2 D_t^{2a}(y(t)) + (2t^{2-a} - (1-a)t^{-a}(t^2+1))(1+t^2) D_t^a(y(t)) + 4n^2 t^{2-2a} y(t) = 0, \; t>0, \tag{2}$$

where D_t^a is a conformable fractional derivative of f with respect to t of order $0 < a \leq 1$ and $D_t^{2a}(y(t)) = D_t^a(D_t^a(y(t)))$.

For $a = 1$ the equation (1) comes down to the classical Sturm-Liouville differential equation of the second order

$$(t^2+1)y''(t) - 2(2n-1)ty'(t) + 2n(2n-1)y(t) = 0, \tag{3}$$

since (2) comes down to

$$(t^2+1)^2 y''(t) + 2t(t^2+1)y'(t) + 4n^2 y(t) = 0. \tag{4}$$

This paper is organized as follows. In Section 2, we stated definitions and assertions used throughout the study. In Section 3, we considered conformable fractional Sturm-Liouville equations (1) and (2) and proved that special polynomials F_n, $n \in \mathbb{N}$, are solutions of the equation (1), since functions $f_0 = 1$, f_n, $n \in \mathbb{N}$, are solutions of equation (2) ([7], [13]) . Through Theorem 3.4 and Theorem 3.5 we connected Mittag-Lefler functions (T_{pj} and $H_{pj}, j = 0, 1, \ldots, p-1, \; p \in \mathbb{N}$) with special polynomials F_n, $n \in \mathbb{N}_0$. At the end of this section we calculate the conformable fractional derivative and the conformable fractional Laplace transform of functions T_{pj} and $H_{pj}, j = 0, 1, \ldots, p-1, \; p \in \mathbb{N}$. Using the conformable fractional Laplace transform we solved the conformable fractional differential equation whose solutions are functions $H_{pj}, j = 0, 1, \ldots, p-1, \; p \in \mathbb{N}$. In the last section (Section 4), we gave some more useful summation formulas for special polynomials F_n, $n \in \mathbb{N}_0$ and solved an open problem from the paper [4].

2. Preliminaries

We employ the following notation: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of positive integers, real and complex numbers, respectively. Also, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a given complex number z, we denote with $\Re(z)$ the real and with $\Im(z)$ the imaginary part. By $L^2(\mathbb{R})$ we denote the space of a square integrable functions. A Laplace transform of a function f is denoted by $\mathcal{L}(f(t))(s) = F(s) = \int_0^{+\infty} f(t)e^{st}dt,\ s \in \mathbb{C}$.

2.1. One class of special polynomials and special functions.

Special polynomials $F_n(t), n \in \mathbb{N}$, introduced in [7] and [13], are defined by:

$$F_0(t) = 1, \quad F_{2n}(t) = \Re((t-i)^{2n}) = \sum_{k=0}^{n} (-1)^{n+k} \binom{2n}{2k} t^{2k}$$

$$F_{2n-1}(t) = \Im((t-i)^{2n}) = \sum_{k=1}^{n} (-1)^{n+k+1} \binom{2n}{2k-1} t^{2k-1}.$$

It is proved in [13] that these polynomials are solutions of the Sturm-Liouville differential equation (3).

Theorem 2.1. [8] *Polynomials $F_n(x)$, $n \in \mathbb{N}_0$ satisfy ($\forall x, t \in \mathbb{R}$):*

$$\sum_{n=0}^{\infty} \frac{F_{2n}(x)}{(2n)!} t^{2n} = \cos t \cdot \cosh(xt), \qquad \sum_{n=1}^{\infty} \frac{F_{2n-1}(x)}{(2n)!} t^{2n} = -\sin t \cdot \sinh(xt),$$

$$\sum_{n=0}^{\infty} \frac{F_{2n}(x)}{n!} t^{n} = \exp(x^2 t - t)\cos(2xt), \qquad \sum_{n=1}^{\infty} \frac{F_{2n-1}(x)}{n!} t^{2} = -\exp(x^2 t - t)\sin(2xt).$$

Using polynomials $F_n(t)$, $n \in \mathbb{N}$, special functions are defined as follows [13]:

$$f_0(t) = 1, \quad f_{2n-1}(t) = (-1)^{n-1} \frac{F_{2n-1}(t)}{(t^2+1)^n}, \quad f_{2n}(t) = (-1)^n \frac{F_{2n}(t)}{(t^2+1)^n},\ n \in \mathbb{N}. \tag{5}$$

Notice that

$$f_{2n-1}(t) = \sin(2n \arctan(t)), \quad f_{2n}(t) = \cos(2n \arctan(t)),\ n \in \mathbb{N}. \tag{6}$$

It is proved in [13] that functions (6) are solutions of the Sturm-Liouville differential equation (4) and form a basis of an $L^2(\mathbb{R})$ space, with respect to the weight function $\omega(t) = \frac{1}{1+t^2}$. Also, in [13] it is shown that these function are utilized to obtain one class of plane curves with the arc length parametrization.

2.2. Functions T_{pj} and H_{pj} as a special case of Mittag-Leffler functions.

Definition 2.1. [4] The functions $T_{pj}, H_{pj} : \mathbb{R} \to \mathbb{R},\ j = 0, 1, 2, \cdots, p-1,\ p \in \mathbb{N}$, are defined as follows:

$$T_{pj}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!}, \quad H_{pj}(t) = \sum_{n=0}^{\infty} \frac{t^{pn+j}}{(pn+j)!}.$$

Theorem 2.2. [4] *For each $t \in \mathbb{R}$, we have*

$$\begin{aligned} T'_{p0}(t) &= -T_{pp-1}(t) & H'_{p0}(t) &= H_{pp-1}(t) \\ T'_{p1}(t) &= T_{p0}(t) & H'_{p1}(t) &= H_{p0}(t) \\ &\vdots & &\vdots \\ T'_{pp-1}(t) &= T_{pp-2}(t) & H'_{pp-1}(t) &= H_{pp-2}(t). \end{aligned}$$

Example 2.2. [4] For each $t \in \mathbb{R}$, we have

$$\begin{aligned} T_{10}(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} = e^{-t} & H_{10}(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^{t} \\ T_{20}(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} = \cos t & T_{21}(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sin t \\ H_{20}(t) &= \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \cosh t & H_{21}(t) &= \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = \sinh t. \end{aligned}$$

Theorem 2.3. [4] *For each $t \in \mathbb{R}$, we have*

$$T_{40}(t) = \cos(\frac{\sqrt{2}}{2}t) \cdot \cosh(\frac{\sqrt{2}}{2}t).$$

Theorem 2.4. [4] *For each $t \in \mathbb{R}$, we have*

$$T_{30}^3(t) - T_{31}^3(t) + T_{32}^3(t) + 3T_{30}(t)T_{31}(t)T_{32}(t) = 1$$
$$H_{30}^3(t) + H_{31}^3(t) + H_{32}^3(t) - 3H_{30}(t)H_{31}(t)H_{32}(t) = 1.$$

Theorem 2.5. [4] *Let p be a prime number and $s \in \mathbb{C}$, then we have*

$$\begin{aligned} \mathcal{L}(T_{pj}(at))(s) &= \frac{s^{p-j-1}a^j}{s^p + a^p}, j = 0, 1, ..., p-1, p \geq 3, \\ \mathcal{L}(H_{pj}(at))(s) &= \frac{s^{p-j-1}a^j}{s^p - a^p}, j = 0, 1, ..., p-1, p \geq 3 \\ \mathcal{L}(\frac{p}{t}(1 - T_{p0}(at)))(s) &= \ln(1 + \frac{a^p}{s^p}) \\ \mathcal{L}(\frac{p}{t}(1 - H_{p0}(at)))(s) &= \ln(1 - \frac{a^p}{s^p}) \\ \mathcal{L}(tT_{p0}(at))(s) &= \frac{s^{2p-2} - (p-1)a^p s^{p-1}}{(s^p + a^p)}. \end{aligned}$$

Using Theorem 2.5 one can solve ordinary differential equations easily. We will demonstrate it on the following two examples.

Example 2.3. The solution of the differential equation

$$y'''(t) - y(t) - 1 = 0, \qquad y(0) = 1 \quad , \quad y'(0) = 2 \quad , \quad y''(0) = 4 \tag{7}$$

is

$$y(t) = 2H_{30}(t) + 2H_{31}(t) + 4H_{32}(t) - 1.$$

If we apply the Laplace transform on (7) we obtain

$$\mathcal{L}(y'''(t)-y(t)-1)(s) = s^3\mathcal{L}(y(t))(s)-s^2y(0)-sy'(0)-y''(0)-\mathcal{L}(y(t))(s)-\mathcal{L}(1)(s) = 0$$

from which it follows

$$\mathcal{L}(y(t))(s) = \frac{s^2+2s+4+\frac{1}{s}}{(s^3-1)} = \frac{s^2}{(s^3-1)} + \frac{2s}{(s^3-1)} + \frac{4}{(s^3-1)} + \frac{1}{s(s^3-1)}.$$

From Theorem 2.5 we have

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(\frac{s^2}{(s^3-1)}) + \mathcal{L}^{-1}(\frac{2s}{(s^3-1)}) + \mathcal{L}^{-1}(\frac{4}{(s^3-1)}) + \mathcal{L}^{-1}(\frac{1}{s(s^3-1)}) \\ &= H_{30}(t) + 2H_{31}(t) + 4H_{32}(t) + H_{30}(t) - 1. \end{aligned}$$

Example 2.4. The solution of the differential equation

$$y^{(5)}(t) - y(t) - 5 = 0, \quad y(0) = 1,\ y'(0) = 2,\ y''(0) = 4,\ y'''(0) = 6,\ y^{(4)}(0) = 8$$

is

$$y(t) = 6H_{30}(t) + 2H_{51}(t) + 4H_{52}(t) + 6H_{53}(t) + 8H_{54}(t) - 5.$$

Similarly, like in example 2.3 we have

$$\begin{aligned} \mathcal{L}(y(t))(s) &= \frac{s^4+2s^3+4s^2+6s+8+\frac{5}{s}}{(s^5-1)} \\ &= \frac{s^4}{(s^5-1)} + \frac{2s^3}{(s^5-1)} + \frac{4s^2}{(s^5-1)} + \frac{6s}{(s^5-1)} + \frac{8}{(s^5-1)} + \frac{5}{s(s^5-1)}. \end{aligned}$$

From Theorem 2.5 we obtain a result

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(\frac{s^4}{(s^5-1)}) + \mathcal{L}^{-1}(\frac{2s^3}{(s^5-1)}) + \mathcal{L}^{-1}(\frac{4s^2}{(s^5-1)}) + \mathcal{L}^{-1}(\frac{6s}{(s^5-1)}) \\ &\quad + \mathcal{L}^{-1}(\frac{8}{(s^5-1)}) + \mathcal{L}^{-1}(\frac{5}{s(s^5-1)}) \\ &= H_{50}(t) + 2H_{51}(t) + 4H_{52}(t) + 6H_{53}(t) + 8H_{54}(t) + 5H_{50}(t) - 5. \end{aligned}$$

2.3. A conformable fractional derivative.

Definition 2.5. [11] The CFD of $y : [0, \infty) \to \mathbb{R}$ with respect to t of order a is defined

$$D_t^a(y(t)) = \lim_{\varepsilon \to 0} \frac{y(t+\varepsilon t^{1-a}) - y(t)}{\varepsilon}, \quad \text{for all} \quad t > 0,\ 0 < a \leq 1. \tag{8}$$

Remark 2.6. If y is a differentiable then

$$D_t^a(y(t)) = t^{1-a}y'(t), \quad \text{for all} \quad t > 0,\ 0 < a \leq 1, \tag{9}$$

where by $'$ we denote the classical derivative.

Theorem 2.6. [10] *Let* $0 < a \leq 1$ *and* $y(t)$ *and* $\tilde{y}(t)$ *be a-conformable differentiable at a point* $t > 0$*, then:*

(i) $D_t^a(c) = 0$, *where c is a constant,*
(ii) $D_t^a(t^c) = \alpha t^{c-a}$, *for all* $c \in \mathbb{R}$,
(iii) $D_t^a(\alpha y(t) + \beta\tilde{y}(t)) = \alpha D_t^a(y(t)) + \beta D_t^a(\tilde{y}(t))$, *for all* $\alpha, \beta \in \mathbb{R}$,
(iv) $D_t^a(y(t)\tilde{y}(t)) = D_t^a(y(t))\tilde{y}(t) + y(t)D_t^a(\tilde{y}(t))$,
(v) $D_t^a\left(\dfrac{y(t)}{\tilde{y}(t)}\right) = \dfrac{D_t^a(y(t))\tilde{y}(t) - y(t)D_t^a(\tilde{y}(t))}{\tilde{y}^2(t)}, \quad \tilde{y}(t) \neq 0.$

Theorem 2.7. [10] *Suppose that the function* $y : (0, +\infty) \to \mathbb{R}$ *is classical and conformable differentiable. The conformable derivative of the function* $y \circ \tilde{y}$ *is*

$$D_t^a(y \circ \tilde{y})(t) = t^{1-a}y'(\tilde{y}(t))\tilde{y}'(t).$$

Definition 2.7. [1] Let $0 < a \leq 1$ and $y : [0, +\infty) \to \mathbb{R}$. The fractional Laplace transform of the function y of an order a is defined by

$$\mathcal{L}_a(y(t))(s) = \int_0^{+\infty} e^{-s\frac{t^a}{a}} y(t)t^{a-1}dt, \quad s \in \mathbb{C}.$$

Lemma 2.8. [1] *Let* $y : [0, \infty) \to \mathbb{R}$ *be a function such that* $\mathcal{L}_a(y(t))(s)$ *exists. Then*

$$\mathcal{L}_a(y(t))(s) = \mathcal{L}(y((at)^{\frac{1}{a}}))(s).$$

Theorem 2.9. *([3],[6]) Let* $y : [0, +\infty) \to \mathbb{R}$ *be a given function,* $0 < a \leq 1$ *and* $s > 0$*. Then*

$$\mathcal{L}_a(D_t^a y(t))(s) = s\mathcal{L}_a(y(t))(s) - y(0),$$
$$\mathcal{L}_a(D_t^{2a} y(t))(s) = s^2\mathcal{L}_a(y(t))(s) - D_t^a y(0) - sy(0).$$

Theorem 2.10. [6] *Let* $y : [0, +\infty) \to \mathbb{R}$ *be a continuous real valued differentiable function and* $0 < a \leq 1$*, then for all* $n \in \mathbb{N}$ *and* $s > 0$*:*

$$\mathcal{L}_a(D_t^{na} y(t))(s) = s^n\mathcal{L}_a(y(t))(s) - \sum_{j=0}^{n-1} s^j D_t^{(n-j-1)a} y(0).$$

3. Main results

Proposition 3.1. *Polynomials* $F_{2n-1}(t)$ *and* $F_{2n}(t)$*,* $n \in \mathbb{N}$*,* $t > 0$*, satisfy:*

$$(t^2 + 1)D_t^a(F_{2n-1}(t)) = 2nt^{1-a}(F_{2n}(t) + tF_{2n-1}(t)) \tag{10}$$

and

$$(t^2 + 1)D_t^a(F_{2n}(t)) = 2nt^{1-a}(tF_{2n}(t) - F_{2n-1}(t)). \tag{11}$$

Proof. Notice that

$$F_{2n}(t) + iF_{2n-1}(t) = (t-i)^{2n}. \tag{12}$$

From Remark 2.6 is

$$D_t^a((t-i)^{2n}) = t^{1-a}((t-i)^{2n})' = 2nt^{1-a}(t-i)^{2n-1}$$

and from Theorem 2.6 it follows

$$D_t^a(F_{2n}(t)) + iD_t^a(F_{2n-1}(t)) = 2nt^{1-a}(t-i)^{2n-1}. \tag{13}$$

By multiplying (13) with $t-i$ and taking the real and imaginary parts we obtain the system

$$\begin{aligned} tD_t^a(F_{2n}(t)) + D_t^a(F_{2n-1}(t)) &= 2nt^{1-a}F_{2n}(t) \\ -D_t^a(F_{2n}(t)) + tD_t^a(F_{2n-1}(t)) &= 2nt^{1-a}F_{2n-1}(t) \end{aligned} \tag{14}$$

from which (10) and (11) follow. □

Theorem 3.2. *Polynomials $F_n(t)$, $n \in \mathbb{N}$, $t > 0$, are solutions of the equation (1).*

Proof. We will prove the assertion only for polynomials $F_{2n-1}(t)$, since the proof for polynomials $F_{2n}(t)$, $n \in \mathbb{N}$, is the same. Applying a CDF of order a on (10) gives

$$\begin{aligned} (t^2+1)D_t^a(D_t^a(F_{2n-1}(t))) + 2t^{2-a}(1-n)D_t^a(F_{2n-1}(t)) - 2nt^{2-2a}(2-a)F_{2n-1}(t) \\ = 2n\left((1-a)t^{1-2a}F_{2n}(t) + t^{1-a}D_t^a(F_{2n}(t)\right). \end{aligned}$$

From (10) and (14) we obtain

$$\begin{aligned} 2n\left((1-a)t^{1-2a}F_{2n}(t) + t^{1-a}D_t^a(F_{2n}(t)\right) \\ = \left(2nt + (1-a)t^{-a}(1+t^2)\right)D_t^a(F_{2n-1}(t)) - 4n^2(2-a)t^{2-2a}F_{2n-1}(t) \end{aligned}$$

from which it follows

$$\begin{aligned} (t^2+1)^2D_t^{2a}(F_{2n-1}(t)) + \left(2t^{2-a}(1-n) - 2nt - (1-a)t^{-a}(1+t^2)\right)D_t^a(F_{2n-1}(t)) \\ + 2nt^{2-2a}(2-a)(2n-1)F_{2n-1}(t) = 0. \end{aligned}$$

□

Theorem 3.3. *Functions $f_n(t)$, $t \in (\tan\frac{k\pi}{n}, \tan(\frac{\pi}{4n} + \frac{k\pi}{n}))$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, are solutions of the equation (2).*

Proof. We will prove the assertion only for functions $f_{2n-1}(t)$, since the proof for functions $f_{2n}(t)$, $n \in \mathbb{N}$, is the same. Applying a CDF of order a on (6), by the use of Theorem 2.7 we obtain

$$(t^2+1)D_t^a(f_{2n-1}(t)) - 2nt^{1-a}f_{2n}(t) = 0 \tag{15}$$

and

$$(t^2+1)D_t^a(f_{2n}(t)) + 2nt^{1-a}f_{2n-1}(t) = 0. \tag{16}$$

Applying a CDF of order a on (15), using Theorem 2.6 we obtain

$$(t^2+1)D_t^a(f_{2n-1}(t))+2t^{2-a}D_t^a(f_{2n}(t))-2n((1-a)t^{1-2a}f_{2n}(t)+t^{1-a}D_t^a(f_{2n}(t))=0.$$

Then, (15) and (16) gives assertion. □

Through the following theorems we will establish the connection between functions T_{pj}, H_{pj} and special polynomials F_n.

Theorem 3.4. *For functions $T_{20}(t), T_{21}(t), T_{40}(t)$, $T_{42}(t)$ and polynomials F_n holds:*

$$T_{40}(t)=\sum_{n=0}^{\infty}\frac{F_{2n}(1)}{2^n(2n)!}t^{2n} \qquad T_{42}(t)=-\sum_{n=1}^{\infty}\frac{F_{2n-1}(1)}{2^n(2n)!}t^{2n}$$

$$T_{20}(t)=\sum_{n=0}^{\infty}\frac{F_{2n}(1)}{2^n n!}t^{n} \qquad T_{21}(t)=-\sum_{n=1}^{\infty}\frac{F_{2n-1}(1)}{2^n n!}t^{n}$$

Proof. Since $F_{2n}(1)=2^n\cos\frac{n\pi}{2}$ and $F_{2n-1}(1)=-2^n\sin\frac{n\pi}{2}$, $n\in\mathbb{N}$, we obtain an assertion. □

Remark 3.1. Using Theorem 2.1 and Theorem 3.4 we can also prove Theorem 2.3:

$$\sum_{n=0}^{\infty}\frac{F_{2n}(1)}{(2n)!}\left(\frac{t}{\sqrt{2}}\right)^{2n}=\cos(\frac{t}{\sqrt{2}})\cdot\cosh(\frac{t}{\sqrt{2}})$$

$$\sum_{n=0}^{\infty}\frac{\cos\frac{n\pi}{2}}{(2n)!}t^{2n}=\cos(\frac{t}{\sqrt{2}})\cdot\cosh(\frac{t}{\sqrt{2}})$$

$$T_{40}(t)=\cos(\frac{t}{\sqrt{2}})\cdot\cosh(\frac{t}{\sqrt{2}})$$

Also, using Theorem 3.4 we can obtain $T_{20}(t)$ and $T_{21}(t)$

$$\sum_{n=0}^{\infty}\frac{F_{2n}(1)}{n!}\left(\frac{t}{2}\right)^{n}=\exp(t-t)\cos t \quad\Longrightarrow\quad T_{20}(t)=\cos t$$

$$\sum_{n=1}^{\infty}\frac{F_{2n-1}(1)}{n!}\left(\frac{t}{2}\right)^{n}=-\exp(t-t)\sin t \quad\Longrightarrow\quad T_{21}(t)=\sin t.$$

Theorem 3.5. *For every $t\in\mathbb{R}$ it holds that*

$$T_{42}(t)=\sin(\frac{t}{\sqrt{2}})\cdot\sinh(\frac{t}{\sqrt{42}}).$$

PROOF. From Theorem 3.4 is

$$\sum_{n=1}^{\infty} \frac{F_{2n-1}(1)}{(2n)!}\left(\frac{t}{\sqrt{2}}\right)^{2n} = -\sin(\frac{t}{\sqrt{2}})\cdot\sinh(\frac{t}{\sqrt{2}})$$

$$-\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{(2n)!}t^{2n} = \sin(\frac{t}{\sqrt{2}})\cdot\sinh(\frac{t}{\sqrt{2}})$$

$$T_{42}(t) = \sin(\frac{t}{\sqrt{2}})\cdot\sinh(\frac{t}{\sqrt{2}})$$

□

Remark 3.2. From Remark 3.1 and Theorem 3.5 we obtain the following equations:

$$T_{40}(t) = T_{20}(\frac{t}{\sqrt{2}})H_{20}(\frac{t}{\sqrt{2}}), \qquad T_{42}(t) = T_{21}(\frac{t}{\sqrt{2}})H_{21}(\frac{t}{\sqrt{2}}).$$

In the following examples we will determine the CFD for functions T_{pj} and H_{pj}:

Example 3.3. Let $t > 0$ and $0 < a \leq 1$, then:

$$\begin{aligned} D_t^a(T_{p0}(qt)) &= -qt^{1-a}T_{pp-1}(qt) & D_t^a(H_{p0}(qt)) &= qt^{1-a}H_{pp-1}(qt)\\ D_t^a(T_{pp-1}(qt)) &= qt^{1-a}T_{pp-2}(qt) & D_t^a(H_{pp-1}(qt)) &= qt^{1-a}H_{pp-2}(qt)\\ D_t^a(T_{pp-2}(qt)) &= qt^{1-a}T_{pp-3}(qt) & D_t^a(H_{pp-2}(qt)) &= qt^{1-a}H_{pp-3}(qt)\\ &\vdots & &\vdots\\ D_t^a(T_{p1}(qt)) &= qt^{1-a}T_{p0}(qt) & D_t^a(H_{p1}(qt)) &= qt^{1-a}H_{p0}(qt). \end{aligned}$$

Using Theorem 2.7 and Theorem 2.2, for $g(t) = T_{p0}(qt)$, we have

$$D_t^a(T_{p0}(qt)) = D_t^a g(t) = t^{1-a}g'(t) = -qt^{1-a}T_{pp-1}(qt).$$

Example 3.4. Let $t > 0$ and $0 < a \leq 1$, then:

$$\begin{aligned} D_t^a(T_{p0}(\frac{1}{a}t^a)) &= -T_{pp-1}(\frac{1}{a}t^a) & D_t^a(H_{p0}(\frac{1}{a}t^a)) &= H_{pp-1}(\frac{1}{a}t^a)\\ D_t^a(T_{pp-1}(\frac{1}{a}t^a)) &= T_{pp-2}(\frac{1}{a}t^a) & D_t^a(H_{pp-1}(\frac{1}{a}t^a)) &= H_{pp-2}(\frac{1}{a}t^a)\\ D_t^a(T_{pp-2}(\frac{1}{a}t^a)) &= T_{pp-3}(\frac{1}{a}t^a) & D_t^a(H_{pp-2}(\frac{1}{a}t^a)) &= H_{pp-3}(\frac{1}{a}t^a)\\ &\vdots & &\vdots\\ D_t^a(T_{p1}(\frac{1}{a}t^a)) &= T_{p0}(\frac{1}{a}t^a) & D_t^a(H_{p1}(\frac{1}{a}t^a)) &= H_{p0}(\frac{1}{a}t^a). \end{aligned}$$

Using Theorem 2.7 and Theorem 2.2, for $T_{pp-1}(\frac{1}{a}t^a)$, we have

$$D_t^a(T_{pp-1}\frac{1}{a}t^a) = D_t^a g(t) = t^{1-a}g'(t) = t^{1-a}t^{a-1}T_{pp-2}(\frac{1}{a}t^a) = T_{pp-2}(\frac{1}{a}t^a)$$

Theorem 3.6. *Let $t > 0$, $0 < a \leq 1$ and p is a prime number. The fractional Laplace of order a for certain functions are:*

$$\begin{aligned}
\mathcal{L}_a(T_{pj}(c\frac{t^a}{a}))(s) &= \frac{s^{p-j-1}c^j}{s^p + c^p}, j = 0, 1, ..., p-1, p \geq 3,\\
\mathcal{L}_a(H_{pj}(c\frac{t^a}{a}))(s) &= \frac{s^{p-j-1}c^j}{s^p - c^p}, j = 0, 1, ..., p-1, p \geq 3\\
\mathcal{L}_a(\frac{cp}{t^a}(1 - T_{p0}(c\frac{t^a}{a})))(s) &= \ln(1 + \frac{c^p}{s^p})\\
\mathcal{L}_a(\frac{ap}{t^a}(1 - T_{p0}(c\frac{t^a}{a})))(s) &= \ln(1 - \frac{c^p}{s^p})\\
\mathcal{L}_a(\frac{t^a}{a}T_{p0}(c\frac{t^a}{a}))(s) &= \frac{s^{2p-2} - (p-1)c^p s^{p-1}}{(s^p + c^p)}.
\end{aligned}$$

Proof. Using Theorem 2.5 and Lemma 2.8 we obtain the proof. □

Using Theorem 3.6 one can solve the conformable fractional differential equations.

Example 3.5. Let $0 < a \leq 1$. The solution of the conformable fractional differential equation

$$\begin{aligned}
&D_t^{5a}(y(t)) - y(t) - 2 = 0,\\
&y(0) = 1, \quad D_t^a y(0) = 3, \quad D_t^{2a} y(0) = 7, \; D_t^{3a} y(0) = 6, \; D_t^{4a} y(0) = 8
\end{aligned} \tag{17}$$

is

$$y(t) = 3H_{50}(\frac{t^a}{a}) + 3H_{51}(\frac{t^a}{a}) + 7H_{52}(\frac{t^a}{a}) + 6H_{53}(\frac{t^a}{a}) + 8H_{54}(\frac{t^a}{a}) - 2.$$

If we apply a generalized conformable fractional Laplace transform on (17), using Theorem 2.10 we obtain

$$\begin{aligned}
0 =& \mathcal{L}_a(D_t^{5a}(y(t)) - y(t) - 2)(s)\\
=& (s^5\mathcal{L}_a(y(t))(s) - s^4 y(0) - s^3 D_t^a y(0) - s^2 D_t^{2a} y(0) - s D_t^{3a} y(0) - D_t^{4a} y(0))\\
&- \mathcal{L}_a(y(t))(s) - 2\mathcal{L}_a(1)(s)
\end{aligned}$$

from which it follows

$$\begin{aligned}
\mathcal{L}_a(y(t))(s) &= \frac{s^4 + 3s^3 + 7s^2 + 6s + 8 + \frac{2}{s}}{(s^5 - 1)}\\
&= \frac{s^4}{(s^5 - 1)} + \frac{3s^3}{(s^5 - 1)} + \frac{7s^2}{(s^5 - 1)} + \frac{6s}{(s^5 - 1)} + \frac{8}{(s^5 - 1)} + \frac{2}{s(s^5 - 1)}.
\end{aligned}$$

By the use of Theorem 3.6 we obtain a solution

$$\begin{aligned} y(t) &= \mathcal{L}_a^{-1}(\frac{s^4}{(s^5-1)}) + \mathcal{L}_a^{-1}(\frac{3s^3}{(s^5-1)}) + \mathcal{L}_a^{-1}(\frac{7s^2}{(s^5-1)}) + \mathcal{L}_a^{-1}(\frac{6s}{(s^5-1)}) \\ &+ \mathcal{L}_a^{-1}(\frac{8}{(s^5-1)}) + \mathcal{L}^{-1}(\frac{5}{s(s^5-1)}) \\ &= H_{50}(\frac{t^a}{a}) + 3H_{51}(\frac{t^a}{a}) + 7H_{52}(\frac{t^a}{a}) + 6H_{53}(\frac{t^a}{a}) + 8H_{54}(\frac{t^a}{a}) + 2H_{50}(\frac{t^a}{a}) - 2. \end{aligned}$$

4. Appendix

Following [8] we give another interesting summation formulas:

Theorem 4.1. *For every $x, t \in \mathbb{R}$ such that $|(x-i)t| < \frac{\pi}{2}$ it holds that*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2^{2n}-1)B_{2n}F_{2n}(x)}{(2n)!} t^{2n} = \frac{xt\tan(xt)(1-\tanh^2(t)) - t\tanh(t)(1+\tan^2(xt))}{1+(\tanh(t)\tan(xt))^2},$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2^{2n}-1)B_{2n}F_{2n-1}(x)}{(2n)!} t^{2n} = \frac{-xt\tanh(t)(1+\tan^2(xt)) - t\tan(xt)(1-\tanh^2(t))}{1+(\tanh(t)\tan(xt))^2}, \tag{18}$$

where B_n are Bernoulli numbers.

Proof. From

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1}$$

we have

$$\tan((x-i)t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2^{2n}-1)B_{2n}}{(2n)!} (x-i)^{2n-1} t^{2n-1}. \tag{19}$$

Since $\tan((x-i)t) = \dfrac{\tan(xt)-\tan(it)}{1+\tan(it)\tan(xt)}$ and $\tan(it) = i\tanh(t)$, we have

$$\begin{aligned} \tan(x-i)t &= \frac{\tan(xt) - i\tanh(t)}{1+i\tanh(t)\tan(xt)} \cdot \frac{1-i\tanh(t)\tan(xt)}{1-i\tanh(t)\tan(xt)} \\ &= \frac{\tan(xt)(1-\tanh^2(t))}{1+(\tanh(t)\tan(xt))^2} - i\frac{\tanh(t)(1+\tan^2(xt))}{1+(\tanh(t)\tan(xt))^2}. \end{aligned}$$

If we multiply equation (19) by $(x-i)$ and take the real and imaginary parts we obtain (18). □

Remark 4.1. We give another proof for Theorem 2.4.

PROOF. Let $\lambda \neq 1, \lambda^3 = 1, \lambda = \frac{-1\pm\sqrt{3}i}{2}, \lambda^2 = \overline{\lambda}, \lambda\overline{\lambda} = 1, \lambda + \overline{\lambda} = -1$, following [4, Remark 4.1] we have

$$e^t e^{\lambda t} e^{\lambda^2 t} = 1.$$

Then

$$\begin{aligned} e^t &= e^{-\lambda t} e^{-\lambda^2 t} \\ &= \left(T_{30}(t) - \lambda T_{31}(t) + \lambda^2 T_{32}(t)\right)\left(T_{30}(t) - \lambda^2 T_{31}(t) + \lambda T_{32}(t)\right) \\ &= \left(T_{30}(t) - \lambda T_{31}(t) + \overline{\lambda} T_{32}(t)\right)\left(T_{30}(t) - \overline{\lambda} T_{31}(t) + \lambda T_{32}(t)\right) \\ &= T_{30}^2(t) + T_{31}^2(t) + T_{32}^2(t) + T_{30}(t)T_{31}(t) - T_{30}(t)T_{32}(t) + T_{31}(t)T_{32}(t). \end{aligned}$$

Now

$$e^{-t} = T_{30}(t) - T_{31}(t) + T_{32}(t)$$

and by $a^3 - b^3 + c^3 + 3abc = (a - b + c)(a^2 + b^2 + c^2 + ab - ac + bc)$ and $e^t e^{-t} = 1$ we obtain the result. □

Theorem 4.2. *(An open problem from [4]) There exists a polynomial of degree $q \geq 5$ such that*

$$P_q(T_{q0}(t), T_{q1}(t), \ldots, T_{qq-1}(t)) = 1.$$

PROOF. We demonstrate a proof when $q = 5$.
Let $\lambda \neq 1, \lambda^5 = 1$,

$$\begin{aligned} \lambda &= \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5}) = a + id & \lambda^2 &= \cos(\frac{4\pi}{5}) + i\sin(\frac{4\pi}{5}) = -b + ic \\ \lambda^3 &= \cos(\frac{6\pi}{5}) + i\sin(\frac{6\pi}{5}) = -b - ic & \lambda^4 &= \cos(\frac{8\pi}{5}) + i\sin(\frac{8\pi}{5}) = a - id \end{aligned}$$

where $a = \frac{-1+\sqrt{5}}{4}, b = \frac{1+\sqrt{5}}{4}, c = \frac{\sqrt{10-2\sqrt{5}}}{4}, d = \frac{\sqrt{10+2\sqrt{5}}}{4}$. By the use [4, Remark 4.1] we have

$$e^t e^{\lambda t} e^{\lambda^2 t} e^{\lambda^3 t} e^{\lambda^4 t} = 1.$$

Then

$$\begin{aligned} e^t &= e^{-\lambda t} e^{-\lambda^2 t} e^{-\lambda^3 t} e^{-\lambda^4 t} \\ &= \left(T_{50}(t) - \lambda T_{51}(t) + \lambda^2 T_{52}(t) - \lambda^3 T_{53}(t) + \lambda^4 T_{54}(t)\right) \\ &\quad \left(T_{50}(t) - \lambda^2 T_{51}(t) + \lambda^4 T_{52}(t) - \lambda T_{53}(t) + \lambda^3 T_{54}(t)\right) \\ &\quad \left(T_{50}(t) - \lambda^3 T_{51}(t) + \lambda T_{52}(t) - \lambda^4 T_{53}(t) + \lambda^2 T_{54}(t)\right) \\ &\quad \left(T_{50}(t) - \lambda^4 T_{51}(t) + \lambda^3 T_{52}(t) - \lambda^2 T_{53}(t) + \lambda T_{54}(t)\right). \end{aligned}$$

Since

$$\begin{aligned}
&\big(T_{50}(t) - \lambda T_{51}(t) + \lambda^2 T_{52}(t) - \lambda^3 T_{53}(t) + \lambda^4 T_{54}(t)\big)\\
=&(T_{50}(t) - (a+id)T_{51}(t) + (-b+ic)T_{52}(t) - (-b-ic)T_{53}(t) + (a-id)T_{54}(t))\\
=&(T_{50}(t) - aT_{51}(t) - bT_{52}(t) + bT_{53}(t) + aT_{54}(t))\\
&+ i\,(-dT_{51}(t) + cT_{52}(t) + cT_{53}(t) - dT_{54}(t))\\
=&A_1 + iB_1
\end{aligned}$$

$$\begin{aligned}
&\big(T_{50}(t) - \lambda^2 T_{51}(t) + \lambda^4 T_{52}(t) - \lambda T_{53}(t) + \lambda^3 T_{54}(t)\big)\\
=&(T_{50}(t) - (-b+ic)T_{51}(t) + (a-id)T_{52}(t) - (a+id)T_{53}(t) + (-b-ic)T_{54}(t))\\
=&(T_{50}(t) + bT_{51}(t) + aT_{52}(t) - aT_{53}(t) - bT_{54}(t))\\
&+ i\,(-cT_{51}(t) - dT_{52}(t) - dT_{53}(t) - cT_{54}(t))\\
=&A_2 + iB_2
\end{aligned}$$

$$\begin{aligned}
&\big(T_{50}(t) - \lambda^3 T_{51}(t) + \lambda T_{52}(t) - \lambda^4 T_{53}(t) + \lambda^2 T_{54}(t)\big)\\
=&(T_{50}(t) - (-b-ic)T_{51}(t) + (a+id)T_{52}(t) - (a-id)T_{53}(t) + (-b+ic)T_{54}(t))\\
=&(T_{50}(t) + bT_{51}(t) + aT_{52}(t) - aT_{53}(t) - bT_{54}(t))\\
&+ i\,(cT_{51}(t) + dT_{52}(t) + dT_{53}(t) + cT_{54}(t))\\
=&A_3 + iB_3
\end{aligned}$$

$$\begin{aligned}
&\big(T_{50}(t) - \lambda^4 T_{51}(t) + \lambda^3 T_{52}(t) - \lambda^2 T_{53}(t) + \lambda T_{54}(t)\big)\\
=&(T_{50}(t) - (a-id)T_{51}(t) + (-b-ic)T_{52}(t) - (-b+ic)T_{53}(t) + (a+id)T_{54}(t))\\
=&(T_{50}(t) - aT_{51}(t) - bT_{52}(t) + bT_{53}(t) + aT_{54}(t))\\
&+ i\,(dT_{51}(t) - cT_{52}(t) - cT_{53}(t) + dT_{54}(t))\\
=&A_4 + iB_4,
\end{aligned}$$

then

$$e^t = (A_1A_2 - B_1B_2)(A_3A_4 - B_3B_4) - (A_1B_2 - B_1A_2)(A_3B_4 - B_3A_4).$$

This is polynomial of degree 4 (by $T_{5j}(t)$, $j = 0, \dots, 4$), so from

$$e^{-t} = T_{50}(t) - T_{51}(t) + T_{52}(t) - T_{53}(t) + T_{54}(t)$$

we conclude that $e^t e^{-t}$ is of degree 5. □

Now we can prove the Theorem 4.2.

PROOF. Solutions of equation $\lambda^q = 1$ are

$$\lambda^k = \cos\frac{2\pi(k-1)}{q} + i\sin\frac{2\pi(k-1)}{q}, \quad k = 1, \dots q.$$

Since, for $\lambda \neq 1$

$$\prod_{k=0}^{q-1} e^{\lambda^k t} = e^{\frac{\lambda^q - 1}{\lambda - 1} t} = 1$$

we have

$$\begin{aligned} e^t &= \prod_{k=1}^{q-1} e^{-\lambda^k t} = \prod_{k=1}^{q-1} \sum_{j=0}^{q-1} (-1)^j \lambda^{kj} T_{qj}(t) \\ &= \prod_{k=1}^{q-1} \sum_{j=0}^{q-1} (-1)^j \left(\cos \frac{2\pi j(k-1)}{q} + i \sin \frac{2\pi j(k-1)}{q} \right) T_{qj}(t). \end{aligned} \tag{20}$$

As conjugate complex solutions of equation $\lambda^q = 1$ come in pairs, following Remark 4.1 and proof of the theorem for $q = 5$ we conclude that a polynomial (20) is a degree of $q - 1$ (by $T_{qj}(t)$, $j = 0, \dots, q-1$). Since

$$e^{-t} = \sum_{j=0}^{q-1} (-1)^j T_{qj}(t),$$

form $e^t \cdot e^{-t} = 1$ we obtain the assertion. □

5. Conclusion

In [13] the authors constructed a novel class of special polynomials and special functions. The uniqueness of those classes lies in the fact that these special polynomials are not orthogonal (the special polynomials most commonly used are orthogonal), but the corresponding class of special functions, with respect to the weight function, is orthonormal. In this paper, through Theorems 3.2 and 3.3, conformable fractional Sturm -Liuoville equations (1) and (2) using novel classes of special polynomials and special functions are solved.

With development the fractional calculus, the importance of the Mittag-Leffler functions was fully understood, since the Mittag-Leffler function arises naturally in the solution of fractional differential equations or fractional integral equations. So, the authors connected the special polynomials introduced in [13] with Mittag-Leffler functions (Theorems 3.4 and 3.5). Also, in the Theorem 3.6 conformable fractional derivatives of certain Mittag-Leffler functions are determined.

In the end of the paper some interesting summation formulas, associated with the novel class of special polynomials, are given and an open problem from [4] is solved.

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