Mathematical Analysis and its Contemporary Applications
Volume 5, Issue 1, 2023, 65–68
doi: 10.30495/maca.2023.1987128.1067
ISSN 2716-9898

Alternative proof of a monotonicity property of certain function

Kwara Nantomah

Abstract. By using L'Hospital's rule for monotonicity, we provide an alternative proof of a monotonicity property of a certain function involving the exponential function. This new approach is very concise.

1. Introduction

Monotonic functions provide remarkable tools for establishing bounds or inequalities and they are frequently encountered in mathematical analysis and other related disciplines.

In the paper [5], the authors proved the following theorem. Subsequently, as applications of the result of the theorem, they establish some inequalities for the function $\frac{e^{-z}}{(1-e^{-z})^2}$ and also establish logarithmic concavity of the function $\frac{z}{e^{az}-e^{(a-1)z}}$ among other things.

Theorem 1.1. *For $z > 0$, the function*

$$\mathcal{K}(z) = \frac{1}{z^2} - \frac{e^{-z}}{(1-e^{-z})^2} \tag{1}$$

is strictly decreasing.

For information on the possible origin of the function $\mathcal{K}(z)$, its connection with other functions, and some other applications, the reader may refer to [3], [4], [5], [7] and the closely related references therein.

2020 *Mathematics Subject Classification.* 26A48.
Key words and phrases. Monotonic function, exponential function, L'Hospital's rule for monotonicity, hyperbolic functions

In this paper, the ultimate goal is to provide an alternative proof of the theorem by using using L'Hospital's rule for monotonicy. Comparatively, the current approach is very simple. We present our proof in the following section.

2. Alternative Proof of Theorem 1.1

In order to construct our proof, we shall require the following Lemmas.

Lemma 2.1 ([6]). *Let $-\infty \leq u < v \leq \infty$ and p and q be continuous functions that are differentiable on (u, v), with $p(u+) = q(u+) = 0$ or $p(v-) = q(v-) = 0$. Suppose that $q(z)$ and $q'(z)$ are nonzero for all $z \in (u, v)$. If $\frac{p'(z)}{q'(z)}$ is increasing (or decreasing) on (u, v), then $\frac{p(x)}{q(x)}$ is also increasing (or decreasing) on (u, v).*

Lemma 2.2. *For $z > 0$, the inequality*

$$z - \sinh(z)\cosh(z) < 0 \tag{2}$$

holds.

Proof. Let $\mathcal{A}(z) = z - \sinh(z)\cosh(z)$ for $z > 0$. Then

$$\begin{aligned}
\mathcal{A}'(z) &= 1 - [\sinh^2(z) + \cosh^2(z)] \\
&= 1 - [2\cosh^2(z) + 1] \\
&= -2\cosh^2(z) \\
&< 0.
\end{aligned}$$

Thus, $\mathcal{A}(z)$ is decreasing. Consequently, for $z > 0$, we have

$$\mathcal{A}(z) < \lim_{z \to 0+} \mathcal{A}(z) = 0$$

which completes the proof of the lemma. □

Proof of Theorem 1.1 . By direct computation, we obtain

$$\begin{aligned}
\mathcal{K}(z) &= \frac{1}{z^2} - \frac{e^{-z}}{(1 - e^{-z})^2} \\
&= \frac{1}{z^2} + \frac{1}{2 - 2\cosh(z)} \\
&= \frac{2 - 2\cosh(z) + z^2}{2z^2(1 - \cosh(z))} = \frac{p_1(z)}{q_1(z)}
\end{aligned}$$

where $p_1(z) = 2 - 2\cosh(z) + z^2$, $q_1(z) = 2z^2(1 - \cosh(z))$ and $p_1(0+) = q_1(0+) = 0$. Then

$$\frac{p_1'(z)}{q_1'(z)} = \frac{z - \sinh(z)}{2z(1 - \cosh(z)) - z^2\sinh(z)} = \frac{p_2(z)}{q_2(z)}$$

where $p_2(z) = z - \sinh(z)$, $q_2(z) = 2z(1 - \cosh(z)) - z^2 \sinh(z)$ and $p_2(0+) = q_2(0+) = 0$. Then

$$\frac{p_2'(z)}{q_2'(z)} = \frac{1 - \cosh(z)}{2 - (z^2 + 2)\cosh(z) - 4z\sinh(z)} = \frac{p_3(z)}{q_3(z)}$$

where $p_3(z) = 1 - \cosh(z)$, $q_3(z) = 2 - (z^2 + 2)\cosh(z) - 4z\sinh(z)$ and $p_3(0+) = q_3(0+) = 0$. Then

$$\frac{p_3'(z)}{q_3'(z)} = \frac{\sinh(z)}{(z^2 + 6)\sinh(z) + 6z\cosh(z)} = \mathcal{B}(z).$$

Then

$$\begin{aligned}
&\left[(z^2 + 6)\sinh(z) + 6z\cosh(z)\right]^2 \mathcal{B}'(z) \\
&= -8z\sinh^2(z) + 6z\cosh^2(z) - 6\sinh(z)\cosh(z) \\
&= -2z\sinh^2(z) + 6z[\cosh^2(z) - \sinh^2(z)] - 6\sinh(z)\cosh(z) \\
&= -2z\sinh^2(z) + 6z - 6\sinh(z)\cosh(z) \\
&= -2z\sinh^2(z) + 6[z - \sinh(z)\cosh(z)] \\
&< 0
\end{aligned}$$

which follows from Lemma 2.2. Thus, the function $\frac{p_3'(z)}{q_3'(z)}$ is strictly decreasing. Therefore, by virtue of Lemma 2.1, the function $\frac{p_1(z)}{q_1(z)}$ is also strictly decreasing and this completes the proof. □

3. Concluding Remarks

In this work, we have provided an alternative proof of a monotonicity property of a certain function involving the exponential function. Comparatively, the current proof is simple and succinct. It has been pointed out in [5] that the function under study is related to

$$\theta(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{e^{-zt}}{t} dt$$

which is the remainder of Binet's first formula for logarithm of the gamma function. We point out here that, the function is also related to

$$\Theta(z) = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{t}{12} \right) \frac{e^{-zt}}{t^2} dt$$

which is the remainder of a Binet-like expression for Barnes G-function (see [2] and a similar expression in [1]).

Acknowledgment

The author wish to thank the anonymous reviewers for careful reading of the paper and for their useful comments. This work was partly supported by the Book and Research Allowance paid to Ghanaian university lecturers.

References

[1] V. S. Adamchik, *Contributions to the Theory of the Barnes Function*, Int. J. Math. Comput. Sci., **9**(1)(2014), 11-30.

[2] J. Choi, *Some mathematical constants*, Appl. Math. Comput., **187**(1)(2007), 122-140.

[3] B-N. Guo, A-Q. Liu and F. Qi, *Monotonicity and logarithmic convexity of three functions involving exponential function*, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., **15**(4)(2008), 387-392.

[4] B-N. Guo and F. Qi, *Two New Proofs of the Complete Monotonicity of a Function Involving the Psi Function*, Bull. Korean Math. Soc., **47**(1)(2010), 103-111.

[5] A-Q. Liu, G-F. Li, B-N. Guo and F. Qi, *Monotonicity and logarithmic concavity of two functions involving exponential function*, Int. J. Math. Ed. Sci. Tech., **39**(5)(2008), 686-691.

[6] I. Pinelis, *L'hospital type Rules for Monotonicity, with Applications*, J. Inequal. Pure Appl. Math., **3**(1)(2002), Art No. 5, 5 pages.

[7] S-Q. Zhang, B-N. Guo and F. Qi, *A Concise Proof For Properties Of Three Functions Involving The Exponential Function*, Appl. Math. E-Notes, **9**(2009), 177-183.

Department of Mathematics, School of Mathematical Sciences, C.K. Tedam University of Technology and Applied Sciences, P. O. Box 24, Navrongo, Upper-East Region, Ghana

Email address: `knantomah@cktutas.edu.gh`,

Received : *January* 2023
Accepted : *February* 2023