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Some fixed point theorems of rational type contraction in complex valued b-metric spaces

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Abstract. The aim of this paper is to prove a common fixed point theorem of rational type contraction in the context of complex valued b-metric spaces and generalizing some results in the existing literature. Finally, We furnish an interesting example in support of our main results.

1. Introduction and Preliminaries

In 2011, Azam et al. [1] defined the concept of a complex valued metric space which is a broadening of the traditional metric space. This line of research has inspired a lot of authors to generalize, extend and improve [1] in various ways, see [2, 5, 7, 8, 10, 13, 16, 17, 18, 19]. Among them, Rao et al. [15] presented the idea of complex valued b-metric space which was more general than the well known complex valued metric spaces [1]. afterwards numbers of papers studied many common fixed point results on b-metric spaces and complex b-metric spaces, for more details, the reader may consult the papers [3, 4, 6, 9, 12, 14].

In this paper, motivated by the above facts, we extend and generalize the results of Hamaizia et al. [11] in complex valued b-metric spaces. We'll need some basic definitions, results, and examples from the literature before we can prove the main results.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.

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Thus $z_1 \precsim z_2$ if one of the following holds:
i) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$;
ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
iii) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
iv) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$.

We will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii), (iii), and (iv) is satisfied; also we will write $z_1 \prec z_2$ if only (iv) is satisfied.

Notice that $0 \precsim z_1 \precnsim z_2$ implies $|z_1| < |z_2|$ and $z_1 \precsim z_2$, $z_2 \prec z_3$ implies $z_1 \prec z_3$. The following definition is recently introduced by Azam et al. [1].

Definition 1.1. Let X be a non empty set, A function $d : X \times X \longrightarrow \mathbb{C}$ is called complex valued metric space if for all $x, y, z \in X$, the following statements hold true:
a) $d(x, y) = 0$ if and only if $x = y$,
b) $d(x, y) = d(y, x)$,
c) $d(x, y) \precsim d(x, z) + d(z, y)$.
The pair (X, d) is called complex valued metric space.

Example 1.2. [17] Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = \exp(ik) |z_1 - z_2|^2,$$

where $k \in \left[0, \frac{\pi}{2}\right]$. Then (X, d) is a complex valued metric space.

Definition 1.3. [15] Let X be a non empty set, $s \geq 1$ a fixed real number, A function $d : X \times X \longrightarrow \mathbb{C}$ is called complex valued b-metric space if for all $x, y, z \in X$, the following statements hold true:
a) $d(x, y) = 0$ if and only if $x = y$
b) $d(x, y) = d(y, x)$,
c) $d(x, y) \precsim s\,[d(x, z) + d(z, y)]$.
The pair (X, d) is called complex valued b-metric space.

Example 1.4. [15] Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = |x - y|^2 + i\,|x - y|^2,$$

for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 1.5. [15] Let (X, d) be a complex valued b-metric space.

i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 < r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) < r\} \subseteq A$.

ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 < r \in \mathbb{C}$, $B(x, r) \cap (A - \{x\}) \neq \phi$.

iii) A subset $A \subseteq X$ is called an open set whenever each element of A is an interior point of a set A.

iv) A subset $A \subseteq X$ is called closed set whenever each limit point of A belongs to A.

v) A sub-basis for Hausdorff topology τ on X is a family

$$F = \{B(x, r) : x \in X \text{ and } 0 < r\}.$$

Definition 1.6. [15] Let (X, d) be a complex valued b-metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$.

i) If for every $c \in C$, with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) < c$, then $\{x_n\}$ is said to be convergent and converges to x. We denote this by $\lim_{n \to +\infty} x_n = x$ or $\{x_n\} \to x$ as $n \to +\infty$.

ii) If for every $c \in C$, with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) < c$ where $m \in \mathbb{N}$, then$\{x_n\}$ is said to be Cauchy sequence.

iii) If every Cauchy sequence in X is convergent in X, then (X, d) is said to be complete complex valued b-metric space.

Lemma 1.1. [15] *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to +\infty$.*

Lemma 1.2. [15] *Let (X, d) be a complex valued b-metric space, and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to +\infty$, where $m \in \mathbb{N}$.*

2. Main results

Now, we are ready to present our main results as follows

Theorem 2.1. *Let (X, d) be a complete complex valued b-metric space with a coefficient $s \geq 1$, and $T : X \to X$ be a mappings on X satisfying the condition*

$$d(Tx, Ty) \precsim ad\,(x, y) + b\frac{d\,(x, Tx)\,d\,(x, Ty) + d\,(y, Ty)\,d\,(y, Tx)}{d\,(x, Ty) + d\,(y, Tx)}, \tag{1}$$

for all, x, y in X and $a, b \geq 0$, $d\,(x, Sy) + d\,(y, Tx) \neq 0$ with $s\,(a + b) < 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X. We define the sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Tx_{2n}, \text{for all } n \in \mathbb{N}.$$

Now, we show that the sequence $\{x_n\}$ is Cauchy

$$\begin{aligned} d\left(x_{2n+1}, x_{2n+2}\right) &= d(Tx_{2n}, Tx_{2n+1}) \\ &\precsim ad\left(x_{2n}, x_{2n+1}\right) + b\frac{d\left(x_{2n}, Tx_{2n}\right) d\left(x_{2n}, Tx_{2n+1}\right)}{d\left(x_{2n}, Tx_{2n+1}\right) + d\left(x_{2n+1}, Tx_{2n}\right)} \\ &\quad + b\frac{d\left(x_{2n+1}, Tx_{2n+1}\right) d\left(x_{2n+1}, Tx_{2n}\right)}{d\left(x_{2n}, Tx_{2n+1}\right) + d\left(x_{2n+1}, Tx_{2n}\right)} \\ &= ad\left(x_{2n}, x_{2n+1}\right) + b\frac{d\left(x_{2n}, x_{2n+1}\right) d\left(x_{2n}, x_{2n+2}\right)}{d\left(x_{2n}, x_{2n+2}\right) + d\left(x_{2n+1}, x_{2n+1}\right)} \\ &\quad + b\frac{d\left(x_{2n+1}, x_{2n+2}\right) d\left(x_{2n+1}, x_{2n+1}\right)}{d\left(x_{2n}, x_{2n+2}\right) + d\left(x_{2n+1}, x_{2n+1}\right)} \\ &= (a+b)\, d\left(x_{2n}, x_{2n+1}\right). \end{aligned}$$

Thus,

$$d\left(x_{2n+1}, x_{2n+2}\right) \precsim (a+b)\, d\left(x_{2n}, x_{2n+1}\right). \tag{2}$$

By using lemma (1.2), implies that

$$\begin{aligned} \left|d\left(x_{2n+1}, x_{2n+2}\right)\right| &\leq \left|(a+b)\, d\left(x_{2n}, x_{2n+1}\right)\right| \\ &\leq (a+b)\left|d\left(x_{2n}, x_{2n+1}\right)\right|. \end{aligned}$$

Since $a+b<1$,

$$\left|d\left(x_{2n+1}, x_{2n+2}\right)\right| \leq (a+b)\left|d\left(x_{2n}, x_{2n+1}\right)\right|. \tag{3}$$

Thus, for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \left|d\left(x_{2n+1}, x_{2n+2}\right)\right| &\leq (a+b)\left|d\left(x_{2n}, x_{2n+1}\right)\right| \leq (a+b)^2\left|d\left(x_{2n-1}, x_{2n-2}\right)\right| \\ &\leq ... \leq (a+b)^{2n+1}\left|d\left(x_1, x_0\right)\right|. \end{aligned} \tag{4}$$

Then, for any $m>n$

$$\begin{aligned} \left|d\left(x_n, x_m\right)\right| &\leq s\left|d\left(x_n, x_{n+1}\right)\right| + s\left|d\left(x_{n+1}, x_m\right)\right| \\ &\leq s\left|d\left(x_n, x_{n+1}\right)\right| + s^2\left|d\left(x_{n+1}, x_{n+2}\right)\right| + s^2\left|d\left(x_{n+2}, x_m\right)\right| \\ &\leq s\left|d\left(x_n, x_{n+1}\right)\right| + s^2\left|d\left(x_{n+1}, x_{n+2}\right)\right| + s^3\left|d\left(x_{n+2}, x_{n+3}\right)\right| \\ &\quad + s^3\left|d\left(x_{n+3}, x_m\right)\right| \\ &\leq s\left|d\left(x_n, x_{n+1}\right)\right| + s^2\left|d\left(x_{n+1}, x_{n+2}\right)\right| + s^3\left|d\left(x_{n+2}, x_{n+3}\right)\right| \\ &\quad + ... + s^{m-n-1}\left|d\left(x_{m-2}, x_{m-1}\right)\right| + s^{m-n}\left|d\left(x_{m-1}, x_m\right)\right|. \end{aligned}$$

By (4), we have

$$\begin{aligned} |d(x_n, x_m)| &\leq s(a+b)^n |d(x_0,x_1)| + s^2 (a+b)^{n+1} |d(x_0,x_1)| \\ &\quad + s^3 (a+b)^{n+2} |d(x_0,x_1)| + ... + s^{m-n-1} (a+b)^{m-2} |d(x_0,x_1)| \\ &\quad + s^{m-n} (a+b)^{m-1} |d(x_0,x_1)| \\ &= \sum_{i=1}^{m-n} s^i (a+b)^{i+n-1} |d(x_0,x_1)|. \end{aligned}$$

Therefore,

$$\begin{aligned} |d(x_n, x_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1} (a+b)^{i+n-1} |d(x_0,x_1)| \\ &= \sum_{p=n}^{m-1} s^p (a+b)^p |d(x_0,x_1)| \\ &\leq \sum_{p=n}^{\infty} [s(a+b)]^p |d(x_0,x_1)| = \frac{[s(a+b)]^p}{1-s(a+b)} |d(x_0,x_1)|. \end{aligned}$$

From which we can deduce that

$$|d(x_n, x_m)| \leq \frac{[s(a+b)]^p}{1-s(a+b)} |d(x_0,x_1)| \to 0 \text{ as } m,n \to +\infty$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to +\infty$. Assume that, there exists $z \in X$ such that

$$|d(u, Tu)| = |z| > 0. \tag{5}$$

Using the triangular inequality and (1), we find

$$\begin{aligned} z &= d(u,Tu) \precsim sd(u, x_{2n+2}) + s|d(x_{2n+2}, Tu)| \\ &= sd(u, x_{2n+2}) + sd(Tu, Sx_{2n+1}) \\ &\precsim sd(u, x_{2n+2}) + sad(u, x_{2n+1}) \\ &\quad + sb\frac{d(u,Tu)\, d(u, Sx_{2n+1}) + d(x_{2n+1}, Sx_{2n+1})\, d(x_{2n+1}, Tu)}{d(u, Sx_{2n+1}) + d(x_{2n+1}, Tu)} \\ &= sd(u, x_{2n+2}) + sad(u, x_{2n+1}) \\ &\quad + sb\frac{d(u,Tu)\, d(u, x_{2n+2}) + d(x_{2n+1}, x_{2n+2})\, d(x_{2n+1}, Tu)}{d(u, x_{2n+2}) + d(x_{2n+1}, Tu)}, \end{aligned}$$

which implies that

$$\begin{aligned} |z| &= |d(u,Tu)| \\ &\leq s|d(u, x_{2n+2})| + sa|d(u, x_{2n+1})| \\ &\quad + sb\frac{|d(u,Tu)|\, |d(u, x_{2n+2})| + |d(x_{2n+1}, x_{2n+2})|\, |d(x_{2n+1}, Tu)|}{|d(u, x_{2n+2}) + d(x_{2n+1}, Tu)|}. \end{aligned} \tag{6}$$

Taking the limit of (6) as $n \to +\infty$, we get that $|z| = |d(u, Tu)| \leq 0$, a contradiction with (5). So $|z| = 0$. Hence,

$$Tu = u.$$

To prove the uniqueness of common fixed point, assume that $v \in X$ be another fixed point of T that is

$$v = Tv$$

It follows that

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\precsim ad(u,v) + b\frac{d(u,Tu)\,d(u,Tv) + d(v,Tv)\,d(v,Tu)}{d(u,Tv) + d(v,Tu)} \\ &= ad(u,v). \end{aligned}$$

Since $a < 1$, we have $d(u,v) = 0$ Thus, T has a unique fixed point in X. □

Theorem 2.2. *Let (X, d) be a complete complex valued b-metric space with $s \geq 1$, and $T, S : X \to X$ be two mappings on X satisfying the condition*

$$d(Tx, Sy) \precsim ad(x,y) + b\frac{d(x,Tx)\,d(x,Sy) + d(y,Sy)\,d(y,Tx)}{d(x,Sy) + d(y,Tx)}, \quad (7)$$

for all, x , y in X and $a, b \geq 0$, $d(x,Sy) + d(y,Tx) \neq 0$ with $s(a+b) < 1$. Then T and S have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X. We define the sequence $\{x_n\}$ in X such that

$$\begin{aligned} x_{2n+1} &= Tx_{2n}, \\ x_{2n+2} &= Sx_{2n+1}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Putting $n = 2k$, with $x = x_{2k}$ and $y = x_{2k+1}$ we get

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(Tx_{2k}, Sx_{2k+1}) \\ &\precsim ad(x_{2k}, x_{2k+1}) + b\frac{d(x_{2k}, Tx_{2k})\,d(x_{2k}, Sx_{2k+1})}{d(x_{2k}, Sx_{2k+1}) + d(x_{2k+1}, Tx_{2k})} \\ &\quad + b\frac{d(x_{2k+1}, Sx_{2k+1})\,d(x_{2k+1}, Tx_{2k})}{d(x_{2k}, Sx_{2k+1}) + d(x_{2k+1}, Tx_{2k})} \\ &= ad(x_{2k}, x_{2k+1}) + b\frac{d(x_{2k}, x_{2k+1})\,d(x_{2k}, x_{2k+2})}{d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})} \\ &\quad + b\frac{d(x_{2k+1}, x_{2k+2})\,d(x_{2k+1}, x_{2k+1})}{d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})} \\ &= (a+b)\,d(x_{2k}, x_{2k+1}). \end{aligned}$$

Thus,

$$d(x_{2k+1}, x_{2k+2}) \precsim (a+b)\, d(x_{2k}, x_{2k+1}). \tag{8}$$

If $x_n = x_{n+1}$ for some n, with $n = 2k$ then from (8), we have

$$d(x_{2k+1}, x_{2k+2}) = 0.$$

which implies that $x_{2k+1} = x_{2k+2}$. For $n = 2k+1$, by using the same arguments as in the case $n = 2k$, we get the same result. Continuing in this way we can show that $x_{2k-1} = x_{2k} = x_{2k+1} = ...$ Then, $\{x_n\}$ is a Cauchy sequence. Now assume that $x_{2k} \neq x_{2k+1}$ for all $n \in \mathbb{N}$. First, we want to show that

$$d(x_n, x_{n+1}) \precsim (a+b)\, d(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N} \tag{9}$$

We consider two cases,

case 1. $n = 2k+1$, $k \in \mathbb{N}$. From (8), we have

$$d(x_n, x_{n+1}) \precsim (a+b)\, d(x_{n-1}, x_n), \; n = 2k+1, k \in \mathbb{N} \tag{10}$$

case 2. $n = 2k$, $k \in \mathbb{N}$. From (8), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\precsim (a+b)\, d(x_n, x_{n+1}) \\ &\precsim d(x_n, x_{n+1}) \precsim (a+b)\, d(x_{n-1}, x_n), \; n = 2k, k \in \mathbb{N}. \end{aligned} \tag{11}$$

So, from (10), (11), we conclude that

$$d(x_n, x_{n+1}) \precsim (a+b)\, d(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}.$$

Thus, we obtain that(9) holds. Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence. By using lemma (1.2) and (8) we obtain

$$\begin{aligned} |d(x_n, x_{n+1})| &\leq |(a+b)\, d(x_{n-1}, x_n)| \\ &\leq (a+b)\, |d(x_{n-1}, x_n)|. \end{aligned}$$

Since $a + b < 1$,

$$|d(x_n, x_{n+1})| \leq (a+b)\, |d(x_{n-1}, x_n)|. \tag{12}$$

Thus, for any $m > n$,

$$\begin{aligned} |d(x_n, x_m)| &\leq s\,|d(x_n, x_{n+1})| + s\,|d(x_{n+1}, x_m)| \\ &\leq s\,|d(x_n, x_{n+1})| + s^2\,|d(x_{n+1}, x_{n+2})| + s^2\,|d(x_{n+2}, x_m)| \\ &\leq s\,|d(x_n, x_{n+1})| + s^2\,|d(x_{n+1}, x_{n+2})| + s^3\,|d(x_{n+2}, x_{n+3})| \\ &\quad + s^3\,|d(x_{n+3}, x_m)| \\ &\leq s\,|d(x_n, x_{n+1})| + s^2\,|d(x_{n+1}, x_{n+2})| + s^3\,|d(x_{n+2}, x_{n+3})| \\ &\quad + \cdots + s^{m-n-1}\,|d(x_{m-2}, x_{m-1})| + s^{m-n}\,|d(x_{m-1}, x_m)|. \end{aligned}$$

By (12) , we find

$$\begin{aligned}|d(x_n,x_m)| &\leq s(a+b)^n|d(x_0,x_1)|+s^2(a+b)^{n+1}|d(x_0,x_1)| \\ &+s^3(a+b)^{n+2}|d(x_0,x_1)|+...+s^{m-n-1}(a+b)^{m-2}|d(x_0,x_1)| \\ &+s^{m-n}(a+b)^{m-1}|d(x_0,x_1)| \\ &= \sum_{i=1}^{m-n}s^i(a+b)^{i+n-1}|d(x_0,x_1)|.\end{aligned}$$

Therefore,

$$\begin{aligned}|d(x_n,x_m)| &\leq \sum_{i=1}^{m-n}s^{i+n-1}(a+b)^{i+n-1}|d(x_0,x_1)| \\ &= \sum_{p=n}^{m-1}s^p(a+b)^p|d(x_0,x_1)| \\ &\leq \sum_{p=n}^{\infty}[s(a+b)]^p|d(x_0,x_1)| = \frac{[s(a+b)]^p}{1-s(a+b)}|d(x_0,x_1)|.\end{aligned}$$

So,

$$|d(x_n,x_m)| \leq \frac{[s(a+b)]^p}{1-s(a+b)}|d(x_0,x_1)| \to 0 \text{ as } m,n\to+\infty$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u\in X$ such that $x_n\to u$ as $n\to+\infty$. Then, there exists $z\in X$ such that

$$|d(u,Tu)| = |z| > 0. \tag{13}$$

So, by using the triangular inequality and (7), we receive

$$\begin{aligned}z &= d(u,Tu) \precsim sd(u,x_{2n+2})+s|d(x_{2n+2},Tu)| \\ &= sd(u,x_{2n+2})+sd(Tu,Sx_{2n+1}) \\ &\precsim sd(u,x_{2n+2})+sad(u,x_{2n+1}) \\ &\quad +sb\frac{d(u,Tu)\,d(u,Sx_{2n+1})+d(x_{2n+1},Sx_{2n+1})\,d(x_{2n+1},Tu)}{d(u,Sx_{2n+1})+d(x_{2n+1},Tu)} \\ &= sd(u,x_{2n+2})+sad(u,x_{2n+1}) \\ &\quad +sb\frac{d(u,Tu)\,d(u,x_{2n+2})+d(x_{2n+1},x_{2n+2})\,d(x_{2n+1},Tu)}{d(u,x_{2n+2})+d(x_{2n+1},Tu)},\end{aligned}$$

which implies that

$$\begin{aligned}|z| &= |d(u,Tu)| \\ &\leq s|d(u,x_{2n+2})|+sa|d(u,x_{2n+1})| \\ &\quad +sb\frac{|d(u,Tu)|\,|d(u,x_{2n+2})|+|d(x_{2n+1},x_{2n+2})|\,|d(x_{2n+1},Tu)|}{|d(u,x_{2n+2})+d(x_{2n+1},Tu)|}.\end{aligned} \tag{14}$$

Taking the limit of (14) as $n \to +\infty$, we get that$|z| = |d(u, Tu)| \leq 0$, a contradiction with (13). We conclude $|z| = 0$. Hence $Tu = u$, Similarly, one can also show that $Su = u$.

To achieve uniqueness of common fixed point, let $v \in X$ be another common fixed point of S and T that is

$$v = Tv = Sv.$$

Then,

$$\begin{aligned} d(u,v) &= d(Tu, Sv) \\ &\precsim ad(u,v) + b\frac{d(u,Tu)\,d(u,Sv) + d(v,Sv)\,d(v,Tu)}{d(u,Sv) + d(v,Tu)} \\ &= ad(u,v). \end{aligned}$$

Since $a < 1$, we have

$$d(u,v) = 0.$$

We conclude that T and S have a unique common fixed point in X. □

Theorem 2.3. *Let(X,d) be a complete complex valued b-metric space with a coefficient $s \geq 1$, and $T, S : X \to X$ be two mappings on X satisfying the condition*

$$d(Tx,Sy) \precsim ad(x,y) + b\frac{d(y,Sy)\,[1 + d(x,Tx)]}{1 + d(x,y)} + c\frac{d(y,Sy) + d(y,Tx)}{1 + d(y,Sy)\,d(y,Tx)}, \tag{15}$$

for all, x, y in X and $a, b, c \geq 0$, and $s(a + b + c) < 1$. Then T and S have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Define by induction a sequence $\{x_n\}$ in X such that

$$\begin{aligned} x_{2n+1} &= Tx_{2n}, \\ x_{2n+2} &= Sx_{2n+1}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

By putting $n = 2k$, with $x = x_{2k}$ and $y = x_{2k+1}$ we get

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(Tx_{2k}, Sx_{2k+1}) \\ &\precsim ad(x_{2k}, x_{2k+1}) + b\frac{d(x_{2k+1}, Sx_{2k+1})\,[1 + d(x_{2k}, Tx_{2k})]}{1 + d(x_{2k}, x_{2k+1})} \\ &\quad + c\frac{d(x_{2k+1}, Sx_{2k+1}) + d(x_{2k+1}, Tx_{2k})}{1 + d(x_{2k+1}, Sx_{2k+1})\,d(x_{2k+1}, Tx_{2k})} \\ &= ad(x_{2k}, x_{2k+1}) + b\frac{d(x_{2k+1}, x_{2k+2})\,[1 + d(x_{2k}, x_{2k+1})]}{1 + d(x_{2k}, x_{2k+1})} \\ &\quad + c\frac{d(x_{2k+1}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})}{1 + d(x_{2k+1}, x_{2k+2})\,d(x_{2k+1}, x_{2k+1})} \\ &= ad(x_{2k}, x_{2k+1}) + bd(x_{2k+1}, x_{2k+2}) + cd(x_{2k+1}, x_{2k+2}). \end{aligned}$$

Thus,

$$d(x_{2k+1}, x_{2k+2}) \precsim \frac{a}{1-(b+c)} d(x_{2k}, x_{2k+1}). \tag{16}$$

If $x_n = x_{n+1}$ for some n, with $n = 2k$ then from (16) ,we have $d(x_{2k+1}, x_{2k+2}) = 0$, so that $x_{2k+1} = x_{2k+2}$. For $n = 2k+1$, by using the same arguments as in the case $n = 2k$,, we find the same result. Continuing in this way, we get $x_{2k-1} = x_{2k} = x_{2k+1} = \cdots$. We find that $\{x_n\}$ is a Cauchy sequence. Assume that $x_{2k} \neq x_{2k+1}$ for all $n \in \mathbb{N}$. First, we want to show that

$$d(x_n, x_{n+1}) \precsim \frac{a}{1-(b+c)} d(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N} \tag{17}$$

There are two cases which we have to consider.

case 1. $n = 2k+1$, $k \in \mathbb{N}$. From (16), we have

$$d(x_n, x_{n+1}) \precsim \frac{a}{1-(b+c)} d(x_{n-1}, x_n), \ n = 2k+1, k \in \mathbb{N}. \tag{18}$$

case 2. $n = 2k$, $k \in \mathbb{N}$. For (16), we get

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\precsim \frac{a}{1-(b+c)} d(x_n, x_{n+1}) \\ &\precsim d(x_n, x_{n+1}) \precsim \frac{a}{1-(b+c)} d(x_{n-1}, x_n), \ n = 2k, k \in \mathbb{N}. \end{aligned} \tag{19}$$

So, from (18) , (19) we conclude that

$$d(x_n, x_{n+1}) \precsim \frac{a}{1-(b+c)} d(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}.$$

Thus, we obtain that (17) holds. Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Using lemma (1.2) and (17), we obtain

$$\begin{aligned} |d(x_n, x_{n+1})| &\leq \left| \left(\frac{a}{1-(b+c)} \right) d(x_{n-1}, x_n) \right| \\ &\leq \left(\frac{a}{1-(b+c)} \right) |d(x_{n-1}, x_n)| \end{aligned}$$

Since $a + b < 1$,

$$|d(x_n, x_{n+1})| \leq h |d(x_{n-1}, x_n)|. \tag{20}$$

where $h = \frac{a}{1-(b+c)} < \frac{1}{s} \leq 1$, because $s(a+b+c) < 1$. Thus, for any $m > n, m, n \in \mathbb{N}$,

$$\begin{aligned} |d(x_n, x_m)| &\leq s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)| \\ &\leq s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| + s^2 |d(x_{n+2}, x_m)| \\ &\leq s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| + s^3 |d(x_{n+2}, x_{n+3})| \\ &\quad + s^3 |d(x_{n+3}, x_m)| \\ &\leq s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| + s^3 |d(x_{n+2}, x_{n+3})| \\ &\quad + \ldots + s^{m-n-1} |d(x_{m-2}, x_{m-1})| + s^{m-n} |d(x_{m-1}, x_m)|. \end{aligned}$$

By (20), we get

$$\begin{aligned}|d(x_n,x_m)| &\leq s(h)^n|d(x_0,x_1)|+s^2(h)^{n+1}|d(x_0,x_1)|\\&+s^3(h)^{n+2}|d(x_0,x_1)|+...+s^{m-n-1}(h)^{m-2}|d(x_0,x_1)|\\&+s^{m-n}(h)^{m-1}|d(x_0,x_1)|\\&=\sum_{i=1}^{m-n}s^i(h)^{i+n-1}|d(x_0,x_1)|.\end{aligned}$$

Then,

$$\begin{aligned}|d(x_n,x_m)| &\leq \sum_{i=1}^{m-n}s^{i+n-1}(h)^{i+n-1}|d(x_0,x_1)|\\&=\sum_{p=n}^{m-1}s^p(h)^p|d(x_0,x_1)|\\&\leq\sum_{p=n}^{\infty}[s(h)]^p|d(x_0,x_1)|=\frac{[s(h)]^p}{1-s(h)}|d(x_0,x_1)|\end{aligned}$$

So,

$$|d(x_n,x_m)|\leq\frac{[s(h)]^p}{1-s(h)}|d(x_0,x_1)|\to 0 \text{ as } m,n\to+\infty.$$

Thus, $\{x_n\}$is a Cauchy sequence in X. Since X is complete, there exists $u\in X$ such that $x_n\to u$ as $n\to+\infty$.

By assumption, there exists $z\in X$ such that

$$|d(u,Tu)|=|z|>0. \tag{21}$$

So, by using the triangular inequality and (15), we receive

$$\begin{aligned}z &= d(u,Tu)\precsim sd(u,x_{2n+2})+s|d(x_{2n+2},Tu)|\\&= sd(u,x_{2n+2})+sd(Tu,Sx_{2n+1})\\&\precsim sd(u,x_{2n+2})+sad(u,x_{2n+1})\\&+sb\frac{d(x_{2n+1},Sx_{2n+1})[1+d(u,Tu)]}{1+d(u,x_{2n+1})}+sc\frac{d(x_{2n+1},Sx_{2n+1})+d(x_{2n+1},Tu)}{1+d(x_{2n+1},Sx_{2n+1})d(x_{2n+1},Tu)}\end{aligned}$$

which implies that

$$\begin{aligned}|z| &= |d(u,Tu)|\\&\leq s|d(u,x_{2n+2})|+sa|d(u,x_{2n+1})|\\&+sb\frac{|d(x_{2n+1},Sx_{2n+1})||1+d(u,Tu)|}{|1+d(u,x_{2n+1})|}+sc\frac{|d(x_{2n+1},Sx_{2n+1})|+|d(x_{2n+1},Tu)|}{|1+d(x_{2n+1},Sx_{2n+1})d(x_{2n+1},Tu)|}\end{aligned} \tag{22}$$

Taking the limit of (22) as $n\to+\infty$, we get that$|z|=|d(u,Tu)|\leq sc|d(u,Tu)|$, a contradiction since $sc<1$. It follows $|z|=0$. Hence $Tu=u$.

On similar steps, we get

$$|d(u, Su)| \leq s\,(b+c)\,|d(u, Su)|\,.$$

Since $s\,(b+c) < 1$, $|d(u, Su)| = 0$ thus $Su = u$. To prove the uniqueness of fixed point, let $v \in X$ be another common fixed point of S and T, it becomes

$$v = Tv = Sv$$

Then

$$\begin{aligned} d\,(u, v) &= d\,(Tu, Sv) \\ &\precsim ad\,(u, v) + b\frac{d\,(v, Sv)\,[1 + d\,(u, Tu)]}{1 + d\,(u, v)} + c\frac{d\,(v, Sv) + d\,(v, Tu)}{1 + d\,(v, Sv)\,d\,(v, Tu)}, \\ &= (a+c)\,d\,(u, v)\,. \end{aligned}$$

Since $0 < a+c < 1$, we have $d\,(u, v) = 0$. Thus, the maps T and S have a unique common fixed point in X. This completes the proof.

□

The following example illustrates the result of 2.1.

Example 2.1. Let $X = [0, 1]\,.$ Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d\,(x, y) = 3\left\{|x-y|^3 + i\,|x-y|^3\right\},$$

for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 4$ To verify that (X, d) is a complete complex valued b-metric space with $s = 4$, it is enough to verify the triangular inequality condition:

$$\begin{aligned} \frac{1}{3}d\,(x, y) &= |x-y|^3 + i\,|x-y|^3 \\ &= |x-y+z-z|^3 + i\,|x-y+z-z|^3 \\ &\preccurlyeq 2^2\left(|x-z|^3 + |z-y|^3\right) + i2^2\left(|x-z|^3 + |z-y|^3\right) \\ &\preccurlyeq 4\left[\left(|x-z|^3 + i\,|x-z|^3\right) + \left(|z-y|^3 + i\,|z-y|^3\right)\right] \\ &= 4\,[d\,(x, z) + d\,(z, y)]\,. \end{aligned}$$

Therefore $s = 4$. Now, define $T : X \to X$ as $Tx = \frac{x}{4}$,$Ty = \frac{y}{4}$, for all $x, y \in X$. Then

$$\begin{aligned} d\,(Tx, Ty) &= d\left(\frac{x}{4}, \frac{y}{4}\right) \\ \frac{1}{3}d\,(Tx, Ty) &= \left\{\left|\frac{x}{4} - \frac{y}{4}\right|^3 + i\left|\frac{x}{4} - \frac{y}{4}\right|^3\right\} \\ &= \frac{1}{4}\left\{|x-y|^3 + i\,|x-y|^3\right\} \\ d\,(Tx, Ty) &= \frac{3}{4}d\,(x, y)\,, \end{aligned}$$

Under the condition (1), we have

$$d(Tx,Ty) \precsim \frac{1}{3}d\left(\frac{x}{4},\frac{y}{4}\right)+\frac{1}{4}\frac{d\left(x,\frac{x}{4}\right)d\left(x,\frac{y}{4}\right)+d\left(y,\frac{y}{4}\right)d\left(y,\frac{x}{4}\right)}{d\left(x,\frac{y}{4}\right)+d\left(y,\frac{x}{4}\right)}.$$

Then

$$s\left(a+b\right)=4\left(\frac{1}{4}.\frac{1}{3}\right)=\frac{1}{3}<1.$$

Thus, all the conditions of Theorem 2.1 are satisfied with the coefficients $s=4, a=\frac{1}{3}$ and $b=\frac{1}{4}$. Observe that the point $0\in X$, remains fixed under T and is indeed unique.

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