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T-norms over complex fuzzy subgroups

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Abstract. In this paper, by using t-norms, we define complex fuzzy subgroups and normal complex fuzzy subgroups and investigate some of characteristics of them. Later we introduce and study the intersection and composition of them. Next, we define the concept of normality between two complex fuzzy subgroups under t-norms and obtain some properties of them. Finally, we define the image and the inverse image of them under group homomorphisms.

1. Introduction

Zadeh [38] proposed the fuzzy sets. The idea of fuzzy sets is based on real number system. Buckley [4, 5] introduced the idea of fuzzy complex sets. In Buckley's definition, the representation of fuzzy complex number in the polar form is quite unstable. Ramot et al. [11, 12] proposed a new concept of defining a fuzzy complex set. Group theory has applications in physics, chemistry, and computer science, and even puzzles like Rubik's Cube can be represented using group theory. Rosenfeld [36] introduced fuzzy sets in the realm of group theory and formulated the concepts of fuzzy subgroups of a group. Many authors have worked on fuzzy group theory [9, 10, 37]. Especially, some authors considered the fuzzy subgroups with respect to a t-norm and gave some results [1, 3, 37]. Alsarahead and Ahmad [2] defined the complex fuzzy subgroup and investigate some of its characteristics. The author by using norms, investigated some properties of fuzzy algebraic structures [13]-[35].

In this paper, by using t-norms, we investigate complex fuzzy subgroups of group G. In Section 2, we recall some basic definitions and preliminary results which will

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be needed in the sequel. In Section 3, we define complex fuzzy subgroups of G under t-norm T as $CFST(G)$ and investigate some properties of them. Later, we define the composition of two $\mu_1, \mu_2 \in CFST(G)$ and obtain some of their characteristics. Also we introduce the intersection of two $\mu_1, \mu_2 \in CFST(G)$ and we prove that $\mu_1 \cap \mu_2 \in CFST(G)$. Next, we define the normality of $\mu \in CFST(G)$ as $NCFST(G)$ and we show that if $\mu_1, \mu_2 \in NCFST(G)$, then $\mu_1 \cap \mu_2 \in NCFST(G)$. Finally, we introduce the normality between two $\mu_1, \mu_2 \in CFST(G)$ and investigate some important properties of them. In Section 4, we investigate obtained conceptions by group homomorphism $f : G \to H$. For this if $\mu \in CFST(G)$ and $\nu \in CFST(H)$, then we prove that $f(\mu) \in CFST(H)$ and $f^{-1}(\nu) \in CFST(G)$. Also if $\mu \in NCFST(G)$ and $\nu \in NCFST(H)$, then we prove that $f(\mu) \in NCFST(H)$ and $f^{-1}(\nu) \in NCFST(G)$. Finally, if $\mu_1, \mu_2 \in CFST(G)$ such that $\mu_1 \preccurlyeq \mu_2$, then we show that $f(\mu_1) \preccurlyeq f(\mu_2)$ and if $\mu_1, \mu_2 \in CFST(H)$ such that $\mu_1 \preccurlyeq \mu_2$, then we obtain that $f^{-1}(\mu_1) \preccurlyeq f^{-1}(\mu_2)$.

2. Preliminaries

Definition 2.1. [7] A group is a nonempty set G on which there is a binary operation $(a, b) \to ab$ such that
(1) if a and b belong to G then ab is also in G (closure),
(2) $a(bc) = (ab)c$ for all $a, b, c \in G$ (associativity),
(3) there is an element $e_G \in G$ such that $ae_G = e_G a = a$ for all $a \in G$ (identity),
(4) if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e_G$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a, b \in G$.

Remark 2.2. There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \to a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \to ab$ for which the identity is denoted by e.

Definition 2.3. [8] Let G be an arbitrary group with a multiplicative binary operation and identity e. As fuzzy subset of G, we mean a function from G into $[0, 1]$. The set of all fuzzy subsets of G is called the $[0, 1]$-power set of G and is denoted $[0, 1]^G$.

Definition 2.4. [11] Let X be a nonempty set. A complex fuzzy set A on X is an object having the form $A = \{(x, \mu_A(x))|x \in X\}$, where μ_A denotes the degree of membership function that assigns each element $x \in X$ a complex number $\mu_A(x)$ lies within the unit circle in the complex plane. We shall assume that is $\mu_A(x)$ will be represented by $r_A(x)e^{iw_A(x)}$ where $i = \sqrt{-1}$, and $r : X \to [0, 1]$ and

$w : X \to [0, 2\pi]$. Note that by setting $w(x) = 0$ in the definition above, we return back to the traditional fuzzy subset. Let $\mu_1 = r_1 e^{w_1}$, and $\mu_2 = r_2 e^{w_2}$ be two complex numbers lie within the unit circle in the complex plane. By $\mu_1 \leq \mu_2$, we mean $r_1 \leq r_2$ and $w_1 \leq w_2$.

Definition 2.5. [6] A t-norm T is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0, 1]$.

We say that T is idempotent if for all $x \in [0, 1]$ we have $T(x, x) = x$.

Example 2.6. The basic t-norms are $T_m(x, y) = min\{x, y\}$ and $T_b(x, y) = max\{0, x+y-1\}$ and $T_p(x, y) = xy$, which are called standard intersection, bounded sum and algebraic product respectively.

Lemma 2.1. [1] *Let T be a t-norm. Then*

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

3. T-norms over complex fuzzy subgroups

Definition 3.1. Let G be a group and $\mu : G \to [0, 1]$ be a complex fuzzy set on G. Then $\mu = re^{iw}$ is said to be a complex fuzzy subgroup of G under t-norm T as $CFST(G)$ if the following conditions hold:
(1) $r(xy) \geq T(r(x), r(y))$,
(2) $r(x^{-1}) \geq r(x)$,
(3) $w(xy) \geq \min\{w(x), w(y)\}$,
(4) $w(x^{-1}) \geq w(x)$,
for all $x, y \in G$.

Example 3.2. Let $G = \{0, a, b, c\}$ be the Klein's group. Every element is its own inverse, and the product of any two distinct non-identity elements is the remaining non-identity element. Thus the Klein 4-group admits the elegant presentation $a^2 = b^2 = c^2 = abc = 0$.
Define $r : G \to [0, 1]$ by

$$r(x) = \begin{cases} 0.5 & \text{if } x = a \\ 0.6 & \text{if } x = b \\ 0.7 & \text{if } x = c \\ 0.8 & \text{if } x = 0 \end{cases}$$

and $w : G \to [0, 2\pi]$ by

$$w(x) = \begin{cases} 0.4\pi & \text{if } x = a \\ 0.4\pi & \text{if } x = b \\ 0.5\pi & \text{if } x = c \\ 0.6\pi & \text{if } x = 0. \end{cases}$$

Let $T(a,b) = T_p(a,b) = ab$ for all $a,b \in [0,1]$, then $\mu(x) = r(x)e^{iw(x)} \in CFST(G)$ for all $x \in G$.

Proposition 3.1. *Let $\mu = re^{iw} \in CFST(G)$ such that $T = \min$ be idempotent t-norm. Then*
(1) $\mu(e) \geq \mu(x)$ for all $x \in G$,
(2) $\mu(x^n) \geq \mu(x)$ for all $x \in G$ and $n \geq 1$,
(3) $\mu(x) = \mu(x^{-1})$ for all $x \in G$.

Proof. Let $\mu = re^{iw} \in CFST(G)$ and $x \in G$ and $n \geq 1$ Then
(1)

$$r(e) = r(xx^{-1}) \geq T(r(x), r(x^{-1})) \geq T(r(x), r(x)) = r(x)$$

and

$$w(e) = w(xx^{-1}) \geq \min\{w(x), w(x^{-1})\} \geq \min\{w(x), w(x)\} = w(x),$$

which mean that

$$\mu(e) = r(e)e^{iw(e)} \geq r(x)e^{iw(x)} = \mu(x).$$

(2)

$$r(x^n) = r(\underbrace{xx...x}_{n}) \geq T(\underbrace{r(x), r(x), ..., r(x)}_{n}) = r(x)$$

and

$$w(x^n) = w(\underbrace{xx...x}_{n}) \geq \min\{\underbrace{w(x), w(x), ..., w(x)}_{n}\} = w(x),$$

which yield

$$\mu(x^n) = r(x^n)e^{iw(x^n)} \geq r(x)e^{iw(x)} = \mu(x).$$

(3) $r(x) = r((x^{-1}))^{-1} \geq r(x^{-1}) \geq r(x)$ and so $r(x) = r(x^{-1})$. Also $w(x) = w((x^{-1}))^{-1} \geq w(x^{-1}) \geq w(x)$ and then $w(x) = w(x^{-1})$. Thus $\mu(x) = r(x)e^{iw(x)} = r(x^{-1})e^{iw(x^{-1})} = \mu(x^{-1})$. □

Proposition 3.2. *Let $\mu = re^{iw} \in CFST(G)$ and $x \in G$ such that T be idempotent t-norm. Then $\mu(xy) = \mu(y)$ for every $y \in G$ if and only if $\mu(x) = \mu(e)$.*

Proof. If $\mu(xy) = \mu(y)$, for all $y \in G$, then as $y = e$ so $\mu(x) = \mu(e)$. Conversely, let $\mu(x) = \mu(e)$, then $r(x) = r(e)$ and $w(x) = w(e)$ and from Proposition 3.1(part

1) we get that $r(x) \geq r(y)$ and $r(x) \geq r(xy)$ also $w(x) \geq w(y)$ and $w(x) \geq w(xy)$. Now

$$\begin{aligned} r(xy) &\geq T(r(x), r(y)) \geq T(r(y), r(y)) = r(y) = r(x^{-1}xy) \geq T(x^{-1}, r(xy)) \\ &\geq T(r(x), r(xy)) \geq T(r(xy), r(xy)) = r(xy), \end{aligned}$$

and then $r(xy) = r(y)$. Also

$$\begin{aligned} w(xy) &\geq \min\{w(x), w(y)\} \geq \min\{w(y), w(y)\} = w(y) = w(x^{-1}xy) \\ &\geq \min\{w(x), w(xy)\} \geq \min\{w(xy), w(xy)\} = w(xy), \end{aligned}$$

thus $w(xy) = w(y)$. Therefore

$$\mu(xy) = r(xy)e^{iw(xy)} = r(y)e^{iw(y)} = \mu(y).$$

□

Definition 3.3. Let G be a set and let $\mu_1 = r_1 e^{iw_1}$ and $\mu_2 = r_2 e^{iw_2}$ be two complex fuzzy sets on G. Denote the composition of μ_1 and μ_2 as $\mu_1 \circ \mu_2 = (r_1 \circ r_2)e^{i(w_1 \circ w_2)}$ such that $r_1 \circ r_2 : G \to [0, 1]$ and $w_1 \circ w_2 : G \to [0, 2\pi]$ and define by $(\mu_1 \circ \mu_2)(x) = (r_1 \circ r_2)(x)e^{i(w_1 \circ w_2)(x)}(x)$ such that

$$(r_1 \circ r_2)(x) = \begin{cases} \sup_{x=ab} T(r_1(a), r_2(b)) & \text{if } x = ab \\ 0 & \text{if } x \neq ab \end{cases}$$

and

$$(w_1 \circ w_2)(x) = \begin{cases} \min_{x=ab}\{w_1(a), w_2(b)\} & \text{if } x = ab \\ 0 & \text{if } x \neq ab, \end{cases}$$

We can say that

$$(\mu_1 \circ \mu_2)(x) = \sup_{x=ab} T(r_1(a), r_2(b))e.^{i \min_{x=ab}}\{w_1(a), w_2(b)\}.$$

Proposition 3.3. *Let μ^{-1} be the inverse of μ such that $\mu^{-1}(x) = \mu(x^{-1})$. Then $\mu \in CFST(G)$ if and only if μ satisfies the following conditions:*
(1) $\mu \geq \mu \circ \mu$;
(2) $\mu^{-1} = \mu$.

Proof. Let $x, y, z \in G$ with $x = yz$ and $\mu \in CFST(G)$. Then

$$r(x) = r(yz) \geq T(r(y), r(z)) = (r \circ r)(x)$$

and

$$w(x) = w(yz) \geq \min\{w(y), w(z)\} = (w \circ w)(x),$$

then

$$\mu(x) = r(x)e^{iw(x)} \geq (r \circ r)(x)e^{i(w \circ w)(x)} = (\mu \circ \mu)(x)$$

so $\mu \geq \mu \circ \mu$. Also as Proposition 3.1 (part3), for all $x \in G$ we have $\mu^{-1}(x) = \mu(x^{-1}) = \mu(x)$ and so $\mu^{-1} = \mu$.

Conversely let $\mu \geq \mu \circ \mu$ and $\mu^{-1} = \mu$. We prove that $\mu \in CFST(G)$. As $\mu \geq \mu \circ \mu$ so $r(x) \geq (r \circ r)(x)$ and $w(x) \geq (w \circ w)(x)$ and thus

$$r(yz) = r(x) \geq (r \circ r)(x) = \sup_{x=yz} T(r(y), r(z)) \geq T(r(y), r(z))$$

and

$$w(yz) = w(x) \geq (w \circ w)(x) = \min_{x=yz}\{w(y), w(z)\} \geq \{w(y), w(z)\}.$$

Since $\mu^{-1} = \mu$ so $r^{-1}(x) = r(x)$ and $w^{-1}(x) = w(x)$, $r(x^{-1}) = r^{-1}(x) = r(x)$ and $w(x^{-1}) = w^{-1}(x) = w(x)$. Then $\mu \in CFST(G)$. □

Corollary 3.4. *Let $\mu_1, \mu_2 \in CFST(G)$ and G be commutative group. Then $\mu_1 \circ \mu_2 \in CFST(G)$ if and only if $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$.*

Proof. As $\mu_1, \mu_2 \in CFST(G)$ and $\mu_1 \circ \mu_2 \in CFST(G)$ then from Proposition 3.3 we get that $\mu_1^{-1} = \mu_1$ and $\mu_2^{-1} = \mu_2$ and $(\mu_2 \circ \mu_1)^{-1} = \mu_2 \circ \mu_1$. Then

$$\mu_1 \circ \mu_2 = \mu_1^{-1} \circ \mu_2^{-1} = (\mu_2 \circ \mu_1)^{-1} = \mu_2 \circ \mu_1.$$

Conversely, let $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$ then

$$(\mu_1 \circ \mu_2) \circ (\mu_1 \circ \mu_2) = \mu_1 \circ (\mu_2 \circ \mu_1) \circ \mu_2 = \mu_1 \circ (\mu_1 \circ \mu_2) \circ \mu_2 = (\mu_1 \circ \mu_1) \circ (\mu_2 \circ \mu_2) \leq \mu_1 \circ \mu_2.$$

Also

$$(\mu_1 \circ \mu_2)^{-1} = (\mu_2 \circ \mu_1)^{-1} = \mu_1^{-1} \circ \mu_2^{-1} = \mu_1 \circ \mu_2.$$

Thus Proposition 3.3 gives us that $\mu_1 \circ \mu_2 \in CFST(G)$. □

Definition 3.4. Let $\mu_1 = r_1 e^{iw_1} \in CFST(G)$ and $\mu_2 = r_2 e^{iw_2} \in CFST(G)$. Define the intersection $\mu_1 \cap \mu_2$ as

$$\mu_1 \cap \mu_2 = r_1 e^{iw_1} \cap r_2 e^{iw_2} = (r_1 \cap r_2) e^{i(w_1 \cap w_2)}$$

such that $r_1 \cap r_2 : G \to [0, 1]$ and $w_1 \cap w_2 : G \to [0, 2\pi]$ and for all $x \in G$ define

$$(r_1 \cap r_2)(x) = T(r_1(x), r_2(x))$$

and

$$(w_1 \cap w_2)(x) = \min\{w_1(x), w_2(x)\}.$$

Proposition 3.5. *Let $\mu_1 = r_1 e^{iw_1} \in CFST(G)$ and $\mu_2 = r_2 e^{iw_2} \in CFST(G)$. Then $\mu_1 \cap \mu_2 \in CFST(G)$.*

Proof. (1) Let $g_1, g_2 \in G$. Then

$$\begin{aligned}
(r_1 \cap r_2)(g_1 g_2) &= T(r_1(g_1 g_2), r_2(g_1 g_2)) \\
&\geq T(T(r_1(g_1), r_1(g_2)), T(r_2(g_1), r_2(g_2))) \\
&= T(T(r_1(g_1), r_2(g_1)), T(r_1(g_2), r_2(g_2))) \quad \textit{(Lemma 2.1)} \\
&= T((r_1 \cap r_2)(g_1), (r_1 \cap r_2)(g_2))
\end{aligned}$$

and thus

$$(r_1 \cap r_2)(g_1 g_2) \geq T((r_1 \cap r_2)(g_1), (r_1 \cap r_2)(g_2)).$$

(2) If $g \in G$, then

$$(r_1 \cap r_2)(g^{-1}) = T(r_1(g^{-1}), r_2(g^{-1})) \geq T(r_1(g), r_2(g)) = (r_1 \cap r_2)(g)$$

and so

$$(r_1 \cap r_2)(g^{-1}) \geq (r_1 \cap r_2)(g).$$

(3) Let $g_1, g_2 \in G$. Then

$$\begin{aligned}(w_1 \cap w_2)(g_1 g_2) &= \min\{w_1(g_1 g_2), w_2(g_1 g_2)\} \\ &\geq \min\{\min\{w_1(g_1), w_1(g_2)\}, \min\{w_2(g_1), w_2(g_2)\}\} \\ &= \min\{\min\{w_1(g_1), w_2(g_1)\}, \min\{w_1(g_2), w_2(g_2))\}\} \\ &= \min\{(w_1 \cap w_2)(g_1), (w_1 \cap w_2)(g_2))\}\end{aligned}$$

and so

$$(w_1 \cap w_2)(g_1 g_2) \geq \min\{(w_1 \cap w_2)(g_1), (w_1 \cap w_2)(g_2))\}.$$

(4) Let $g \in G$ so

$$(w_1 \cap w_2)(g^{-1}) = \min\{w_1(g^{-1}), w_2(g^{-1})\} \geq \min\{w_1(g), w_2(g)\} = (w_1 \cap w_2)(g)$$

and then

$$(w_1 \cap w_2)(g^{-1}) \geq (w_1 \cap w_2)(g).$$

Thus from (1)-(4) we give that $\mu_1 \cap \mu_2 \in CFST(G)$. □

Corollary 3.6. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq CFST(G)$ then*

$$\mu = \cap_{i \in I_n} \mu_i \in CFST(G).$$

Definition 3.5. $\mu \in CFST(G)$ is called normal as $NCFST(G)$, if for all $x, y \in G$ we have $\mu(xyx^{-1}) = \mu(y)$.

Proposition 3.7. *Let $\mu_1 = r_1 e^{iw_1} \in NCFST(G)$ and $\mu_2 = r_2 e^{iw_2} \in NCFST(G)$. Then $\mu_1 \cap \mu_2 \in NCFST(G)$.*

Proof. By Proposition 3.5 we will have that $\mu_1 \cap \mu_2 \in CFST(G)$. Let $g_1, g_2 \in G$ then

$$(r_1 \cap r_2)(g_1 g_2 g_1^{-1}) = T(r_1(g_1 g_2 g_1^{-1}), r_2(g_1 g_2 g_1^{-1})) = T(r_1(g_2), r_2(g_2)) = (r_1 \cap r_2)(g_2)$$

and

$$\begin{aligned}(w_1 \cap w_2)(g_1 g_2 g_1^{-1}) &= \min\{w_1(g_1 g_2 g_1^{-1}), w_2(g_1 g_2 g_1^{-1})\} \\ &= \min\{w_1(g_2), w_2(g_2)\} = (w_1 \cap w_2)(g_2)\end{aligned}$$

and thus

$$\begin{aligned}(\mu_1 \cap \mu_2)(g_1 g_2 g_1^{-1}) =&(r_1 \cap r_2)(g_1 g_2 g_1^{-1})e^{i(w_1 \cap w_2)(g_1 g_2 g_1^{-1})}\\ =&(r_1 \cap r_2)(g_2)e^{i(w_1 \cap w_2)(g_2)} = (\mu_1 \cap \mu_2)(g_2)\end{aligned}$$

and therefore $\mu_1 \cap \mu_2 \in NCFST(G)$. □

Corollary 3.8. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq NCFST(G)$, Then $\mu = \cap_{i \in I_n} \mu_i \in NCFST(G)$.*

Definition 3.6. Let $\mu_1 = r_1 e^{iw_1} \in CFST(G)$ and $\mu_2 = r_2 e^{iw_2} \in CFST(G)$ such that $\mu_1 \subseteq \mu_2$. We say that μ_1 is normal of the μ_2, written $\mu_1 \preccurlyeq \mu_2$, if

$$r_1(g_1 g_2 g_1^{-1}) \geq T(r_1(g_2), r_2(g_1)) \; and \; w_1(g_1 g_2 g_1^{-1}) \geq \min\{w_1(g_2), w_2(g_1)\}$$

for all $g_1, g_2 \in G$.

Proposition 3.9. *If T be idempotent, then every $\mu = re^{iw} \in CFST(G)$ will be normal of itself.*

Proof. Let $g_1, g_2 \in G$ and $\mu = re^{iw} \in CFST(G)$. Then

$$\begin{aligned}r(g_1 g_2 g_1^{-1}) &\geq T(r(g_1), r(g_2 g_1^{-1}))\\ &\geq T(r(g_1), T(r(g_2), r(g_1^{-1})))\\ &\geq T(r(g_1), T(r(g_2), r(g_1)))\\ &= T(r(g_2), T(r(g_1), r(g_1)))\\ &= T(r(g_2), r(g_1))\end{aligned}$$

and so

$$r(g_1 g_2 g_1^{-1}) \geq T(r(g_2), r(g_1)).$$

Also

$$\begin{aligned}w(g_1 g_2 g_1^{-1}) &\geq \min\{w(g_1), w(g_2 g_1^{-1}))\}\\ &\geq \min\{w(g_1), \min\{w(g_2), w(g_1^{-1})\}\}\\ &\geq \min\{w(g_1), \min\{w(g_2), w(g_1)))\\ &= \min\{w(g_2), \min\{w(g_1), w(g_1)\}\}\\ &= \min\{w(g_2), w(g_1)\}\end{aligned}$$

and then

$$w(g_1 g_2 g_1^{-1}) \geq \min\{w(g_2), w(g_1)\}.$$

Therefore $\mu = re^{iw} \preccurlyeq \mu = re^{iw}$. □

Proposition 3.10. *Let $\mu_1 = r_1 e^{iw_1} \in NCFST(G)$ and $\mu_2 = r_2 e^{iw_2} \in CFST(G)$ such that T be idempotent. Then $\mu_1 \cap \mu_2 \preccurlyeq \mu_2$.*

Proof. By Proposition 3.5 $(\mu_1 \cap \mu_2) \leq \mu_2$ and $(\mu_1 \cap \mu_2) \in CFST(G)$. Let $g_1, g_2 \in G$ and $\mu_1 \cap \mu_2 = (r_1 \cap r_2)e^{i(w_1 \cap w_2)}$. Then

$$\begin{aligned}
(r_1 \cap r_2)(g_1 g_2 g_1^{-1}) &= T(r_1(g_1 g_2 g_1^{-1}), r_2(g_1 g_2 g_1^{-1})) \\
&= T(r_1(g_2), r_2(g_1 g_2 g_1^{-1})) \\
&\geq T(r_1(g_2), T(r_2(g_1 g_2), r_2(g_1^{-1}))) \\
&\geq T(r_1(g_2), T(r_2(g_1 g_2), r_2(g_1))) \\
&\geq T(r_1(g_2), T(T(r_2(g_1), r_2(g_2)), r_2(g_1))) \\
&= T(r_1(g_2), T(T(r_2(g_1), r_2(g_1)), r_2(g_2))) \\
&= T(r_1(g_2), T(r_2(g_1), r_2(g_2))) \\
&= T(T(r_1(g_2), r_2(g_2)), r_2(g_1)) \\
&= T((r_1 \cap r_2)(g_2), r_2(g_1))
\end{aligned}$$

and thus

$$(r_1 \cap r_2)(g_1 g_2 g_1^{-1}) \geq T((r_1 \cap r_2)(g_2), r_2(g_1)).$$

Also

$$\begin{aligned}
(w_1 \cap w_2)(g_1 g_2 g_1^{-1}) &= \min\{w_1(g_1 g_2 g_1^{-1}), w_2(g_1 g_2 g_1^{-1})\} \\
&= \min\{w_1(g_2), w_2(g_1 g_2 g_1^{-1})\} \\
&\geq \min\{w_1(g_2), \min\{w_2(g_1 g_2), w_2(g_1^{-1})\}\} \\
&\geq \min\{w_1(g_2), \min\{w_2(g_1 g_2), w_2(g_1)\}\} \\
&\geq \min\{w_1(g_2), \min\{\min\{w_2(g_1), w_2(g_2)\}, w_2(g_1)\}\} \\
&= \min\{w_1(g_2), \min\{\min\{w_2(g_1), w_2(g_1))\}, w_2(g_2)\}\} \\
&= \min\{w_1(g_2), \min\{w_2(g_1), w_2(g_2)\}\} \\
&= \min\{\min\{w_1(g_2), w_2(g_2)\}, w_2(g_1)\} \\
&= \min\{(w_1 \cap w_2)(g_2), w_2(g_1)\}
\end{aligned}$$

and then

$$(w_1 \cap w_2)(g_1 g_2 g_1^{-1}) \geq \min\{(w_1 \cap w_2)(g_2), w_2(g_1)\}.$$

Therefore $\mu_1 \cap \mu_2 = (r_1 \cap r_2)e^{i(w_1 \cap w_2)} \preccurlyeq \mu_2$. □

Proposition 3.11. *Let $\mu_1 = r_1 e^{iw_1} \in CFST(G)$ and $\mu_2 = r_2 e^{iw_2} \in CFST(G)$ and $\mu_3 = r_3 e^{iw_3} \in CFST(G)$ and T be idempotent t-norm. If $\mu_1 \preccurlyeq \mu_3$ and $\mu_2 \preccurlyeq \mu_3$, then $\mu_1 \cap \mu_2 \preccurlyeq \mu_3$.*

Proof. As Proposition 3.5 we will have that $\mu_1 \cap \mu_2 \in CFST(G)$ and $\mu_1 \cap \mu_2 \leq \mu_3$. Let $g_1, g_2 \in G$. As $\mu_1 \preccurlyeq \mu_3$ so $r_1(g_1 g_2 g_1^{-1}) \geq T(r_1(g_2), r_3(g_1))$ and

$w_1(g_1g_2g_1^{-1}) \geq \min\{w_1(g_2), w_3(g_1)\}$ and as $\mu_2 \preccurlyeq \mu_3$ so $r_2(g_1g_2g_1^{-1}) \geq T(r_2(g_2), r_3(g_1))$ and $w_2(g_1g_2g_1^{-1}) \geq \min\{w_2(g_2), w_3(g_1)\}$. Now

$$\begin{aligned}(r_1 \cap r_2)(g_1g_2g_1^{-1}) &= T(r_1(g_1g_2g_1^{-1}), r_2(g_1g_2g_1^{-1})) \\ &\geq T(T(r_1(g_2), r_3(g_1)), T(r_2(g_2), r_3(g_1))) \\ &= T(T(r_1(g_2), r_2(g_2)), T(r_3(g_1), r_3(g_1))) \text{ (Lemma 2.1)} \\ &= T(T(r_1(g_2), r_2(g_2)), r_3(g_1)) \\ &= T((r_1 \cap r_2)(g_2), r_3(g_1))\end{aligned}$$

and then

$$(r_1 \cap r_2)(g_1g_2g_1^{-1}) \geq T((r_1 \cap r_2)(g_2), r_3(g_1)).$$

Also

$$\begin{aligned}(w_1 \cap w_2)(g_1g_2g_1^{-1}) &= \min\{w_1(g_1g_2g_1^{-1}), w_2(g_1g_2g_1^{-1})\} \\ &\geq \min\{\min\{w_1(g_2), w_3(g_1)\}, \min\{w_2(g_2), w_3(g_1)\}\} \\ &= \min\{\min\{w_1(g_2), w_2(g_2)\}, \min\{w_3(g_1), w_3(g_1)\}\} \\ &= \min\{\min\{w_1(g_2), w_2(g_2)\}, w_3(g_1)\} \\ &= \min\{(w_1 \cap w_2)(g_2), w_3(g_1)\}\end{aligned}$$

and so

$$(w_1 \cap w_2)(g_1g_2g_1^{-1}) \geq \min\{(w_1 \cap w_2)(g_2), w_3(g_1)\}.$$

Thus $\mu_1 \cap \mu_2 = (r_1 \cap r_2)e^{i(w_1 \cap w_2)} \preccurlyeq \mu_3$. □

Corollary 3.12. *Let $I_n = \{1, 2, ..., n\}$ and $\{\mu_i \mid i \in I_n\} \subseteq CFST(G)$ such that $\{\mu_i \mid i \in I_n\} \preccurlyeq \xi$. Then $\mu = \cap_{i \in I_n} \mu_i \preccurlyeq \xi$.*

4. Investigated obtained conceptions under group homomorphisms

Definition 4.1. Let $f : G \to H$ be a mapping and $\mu_G = r_G e^{iw_G}$ and $\mu_H = r_H e^{iw_H}$ be two complex fuzzy sets on G and H, respectively. Define $f(\mu_G) : H \to [0, 1]$ as

$$f(\mu_G) = f(r_G e^{iw_G}) = f(r_G)e^{if(w_G)},$$

such that for all $h \in H$ we define

$$f(r_G)(h) = \sup\{r_G(g) \mid g \in G, f(g) = h\}$$

and

$$f(w_G)(h) = \sup\{w_G(g) \mid g \in G, f(g) = h\}.$$

Also define $f^{-1}(\mu_H) : G \to [0, 1]$ as

$$f^{-1}(r_H e^{iw_H}) = f^{-1}(r_H)e^{if^{-1}(w_H)}$$

such that for all $g \in G$ we define

$$f^{-1}(r_H e^{iw_H})(g) = r_H(f(g))e^{iw_H(f(g))}.$$

Proposition 4.1. *Let $\mu_G = r_G e^{iw_G} \in CFST(G)$ and $f : G \to H$ be a group homomorphism. Then $f(\mu_G) \in CFST(H)$.*

PROOF. (1) Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$ such that $h_1 = f(g_1)$ and $h_2 = f(g_2)$. Then

$$\begin{aligned} f(r_G)(h_1h_2) &= \sup\{r_G(g_1g_2) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &\geq \sup\{T(r_G(g_1), r_G(g_2)) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &= T(\sup\{r_G(g_1) \mid g_1 \in G, f(g_1) = h_1\}, \sup\{r_G(g_2) \mid g_2 \in G, f(g_2) = h_2\}) \\ &= T(f(r_G)(h_1), f(r_G)(h_2)) \end{aligned}$$

and so

$$f(r_G)(h_1h_2) \geq T(f(r_G)(h_1), f(r_G)(h_2)).$$

(2) Let $h \in H$ and $g \in G$ such that $h = f(g)$. Then

$$\begin{aligned} f(r_G)(h^{-1}) &= \sup\{r_G(g^{-1}) \mid g^{-1} \in G, f(g^{-1}) = h^{-1}\} \\ &\geq \sup\{r_G(g) \mid g \in G, f^{-1}(g) = h^{-1}\} \\ &= \sup\{r_G(g) \mid g \in G, f(g) = h\} \\ &= f(r_G)(h) \end{aligned}$$

and so

$$f(r_G)(h^{-1}) \geq f(r_G)(h).$$

(3) Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$ such that $h_1 = f(g_1)$ and $h_2 = f(g_2)$. Then

$$\begin{aligned} f(w_G)(h_1h_2) &= \sup\{w_G(g_1g_2) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &\geq \sup\{\min\{w_G(g_1), w_G(g_2)\} \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &= \min\{\sup\{w_G(g_1) \mid g_1 \in G, f(g_1) = h_1\}, \sup\{w_G(g_2) \mid g_2 \in G, f(g_2) = h_2\}) \\ &= \min\{f(w_G)(h_1), f(w_G)(h_2)\} \end{aligned}$$

and thus

$$f(w_G)(h_1h_2) \geq \min\{f(w_G)(h_1), f(w_G)(h_2)\}.$$

(4) Let $h \in H$ and $g \in G$ such that $h = f(g)$. As

$$\begin{aligned} f(w_G)(h^{-1}) &= \sup\{w_G(g^{-1}) \mid g^{-1} \in G, f(g^{-1}) = h^{-1}\} \\ &\geq \sup\{w_G(g) \mid g^{-1} \in G, f^{-1}(g) = h^{-1}\} \\ &= \sup\{w_G(g) \mid g \in G, f(g) = h\} \\ &= f(w_G)(h) \end{aligned}$$

so

$$f(w_G)(h^{-1}) \geq f(w_G)(h).$$

Thus (1) - (4) mean that $f(\mu_G) = f(r_G e^{iw_G}) = f(r_G)e^{if(w_G)} \in CFST(H)$. $\square$

Proposition 4.2. *Let $\mu_H = r_H e^{iw_H} \in CFST(H)$ and $f : G \to H$ be a group homomorphism. Then $f^{-1}(\mu_H) \in CFST(G)$.*

Proof. (1) Let $g_1, g_2 \in G$, then

$$\begin{aligned} f^{-1}(r_H)(g_1 g_2) &= r_H(f(g_1 g_2)) \\ &= r_H(f(g_1)f(g_2)) \\ &\geq T(r_H(f(g_1)), r_H(f(g_2))) \\ &= T(f^{-1}(r_H)(g_1), f^{-1}(r_H)(g_2)), \end{aligned}$$

therefore

$$f^{-1}(r_H)(g_1 g_2) \geq T(f^{-1}(r_H)(g_1), f^{-1}(r_H)(g_2)).$$

(2) Let $g \in G$ then

$$f^{-1}(r_H)(g^{-1}) = r_H(f(g^{-1})) = r_H(f^{-1}(g)) \geq r_H(f(g)) = f^{-1}(r_H)(g)$$

and thus

$$f^{-1}(r_H)(g^{-1}) \geq f^{-1}(r_H)(g).$$

(3) Let $g_1, g_2 \in G$ so

$$\begin{aligned} f^{-1}(w_H)(g_1 g_2) &= w_H(f(g_1 g_2)) \\ &= w_H(f(g_1)f(g_2)) \\ &\geq \min\{w_H(f(g_1)), w_H(f(g_2))\} \\ &= \min\{f^{-1}(w_H)(g_1), f^{-1}(w_H)(g_2)\} \end{aligned}$$

and then

$$f^{-1}(w_H)(g_1 g_2) \geq \min\{f^{-1}(w_H)(g_1), f^{-1}(w_H)(g_2))\}.$$

(4) Let $g \in G$ then

$$f^{-1}(w_H)(g^{-1}) = w_H(f^{-1}(g)) \geq w_H(f(g)) = f^{-1}(w_H)(g).$$

Therefore (1)-(4) give us $f^{-1}(r_H e^{iw_H})(g) = r_H(f(g))e^{iw_H(f(g))} \in CFST(G)$. $\square$

Proposition 4.3. *Let $\mu_G = r_G e^{iw_G} \in NCFST(G)$ and $f : G \to H$ be a group homomorphism. Then $f(\mu_G) \in NCFST(H)$.*

PROOF. As Proposition 4.1 we get that $f(\mu_G) \in CFST(H)$. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$ such that $f(g_1) = h_1$ and $f(g_2) = h_2$. Now

$$\begin{aligned} f(r_G)(h_1h_2h_1^{-1}) &= \sup\{r_G(g_1g_2g_1^{-1}) \mid f(g_1g_2g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \sup\{r_G(g_2) \mid f(g_1)f(g_2)f(g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \sup\{r_G(g_2) \mid f(g_1)f(g_2)f^{-1}(g_1) = h_1h_2h_1^{-1}\} \\ &= \sup\{r_G(g_2) \mid f(g_2) = h_2\} \\ &= f(r_G)(h_2). \end{aligned}$$

Also

$$\begin{aligned} f(w_G)(h_1h_2h_1^{-1}) &= \sup\{w_G(g_1g_2g_1^{-1}) \mid f(g_1g_2g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \sup\{w_G(g_2) \mid f(g_1)f(g_2)f(g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \sup\{w_G(g_2) \mid f(g_1)f(g_2)f^{-1}(g_1) = h_1h_2h_1^{-1}\} \\ &= \sup\{w_G(g_2) \mid f(g_2) = h_2\} \\ &= f(w_G)(h_2). \end{aligned}$$

Then

$$f(\mu_G)(h_1h_2h_1^{-1}) = f(r_G)(h_1h_2h_1^{-1})e^{if(w_G)(h_1h_2h_1^{-1})} = f(r_G)(h_2)e^{if(w_G)(h_2)} = f(\mu_G)(h_2)$$

and so $f(\mu_G) \in NCFST(H)$. □

Proposition 4.4. *Let $\mu_H = r_He^{iw_H} \in NCFST(H)$ and $f : G \to H$ be a group homomorphism. Then $f^{-1}(\mu_H) \in NCFST(G)$.*

PROOF. Using Proposition 4.2 we get that $f^{-1}(\mu_H) \in CFST(G)$. Let $g_1, g_2 \in G$ then

$$\begin{aligned} f^{-1}(r_H)(g_1g_2g_1^{-1}) &= r_H(f(g_1g_2g_1^{-1})) \\ &= r_H(f(g_1)f(g_2)f(g_1^{-1})) \\ &= r_H(f(g_1)f(g_2)f^{-1}(g_1)) \\ &= r_H(f(g_2)) \\ &= f^{-1}(r_H)(g_2). \end{aligned}$$

Also

$$\begin{aligned} f^{-1}(w_H)(g_1g_2g_1^{-1}) &= w_H(f(g_1g_2g_1^{-1})) \\ &= w_H(f(g_1)f(g_2)f(g_1^{-1})) \\ &= w_H(f(g_1)f(g_2)f^{-1}(g_1)) \\ &= w_H(f(g_2)) \\ &= f^{-1}(w_H)(g_2). \end{aligned}$$

Thus

$$\begin{aligned} f^{-1}(\mu_H)(g_1g_2g_1^{-1}) &= f^{-1}(r_H)(g_1g_2g_1^{-1})e^{if^{-1}(w_H)(g_1g_2g_1^{-1})} \\ &= f^{-1}(r_H)(g_2)e^{if^{-1}(w_H)(g_2)} \\ &= f^{-1}(\mu_H)(g_2) \end{aligned}$$

and thus $f^{-1}(\mu_H) \in NCFST(G)$. □

Proposition 4.5. *Let $\mu_1 = r_1e^{iw_1} \in CFST(G)$ and $\mu_2 = r_2e^{iw_2} \in CFST(G)$ and $f : G \to H$ be a group homomorphism. If $\mu_1 \preccurlyeq \mu_2$, then $f(\mu_1) \preccurlyeq f(\mu_2)$.*

Proof. We know that $f(\mu_1) = f(r_1)e^{if(w_1)}$ and $f(\mu_2) = f(r_2)e^{if(w_2)}$. By Proposition 4.1 we get that $f(\mu_1) \in CFST(H)$ and $f(\mu_2) \in CFST(H)$. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$ such that $f(g_1) = h_1$ and $f(g_2) = h_2$. Since $\mu_1 \preccurlyeq \mu_2$ so $r_1(g_1g_2g_1^{-1}) \geq T(r_1(g_2), r_2(g_1))$ and $w_1(g_1g_2g_1^{-1}) \geq \min\{w_1(g_2), w_2(g_1)\}$. Now

$$\begin{aligned} f(r_1)(h_1h_2h_1^{-1}) &= \sup\{r_1(g_1g_2g_1^{-1}) \mid f(g_1g_2g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &\geq \sup\{T(r_1(g_2), r_2(g_1)) \mid f(g_1)f(g_2)f(g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \sup\{T(r_1(g_2), r_2(g_1)) \mid f(g_1)f(g_2)f^{-1}(g_1) = h_1h_2h_1^{-1}\} \\ &= T(\sup\{r_1(g_2) \mid f(g_2) = h_2\}, \sup\{r_2(g_1) \mid f(g_1) = h_1\}) \\ &= T(f(r_1)(h_2), f(r_2)(h_1)) \end{aligned}$$

and then

$$f(r_1)(h_1h_2h_1^{-1}) \geq T(f(r_1)(h_2), f(r_2)(h_1)).$$

Also

$$\begin{aligned} f(w_1)(h_1h_2h_1^{-1}) &= \sup\{w_1(g_1g_2g_1^{-1}) \mid f(g_1g_2g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &\geq \sup\{\min\{w_1(g_2), w_2(g_1)\} \mid f(g_1)f(g_2)f(g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \sup\{\min\{w_1(g_2), w_2(g_1)\} \mid f(g_1)f(g_2)f^{-1}(g_1) = h_1h_2h_1^{-1}\} \\ &= \min\{\sup\{w_1(g_2) \mid f(g_2) = h_2\}, sup\{w_2(g_1) \mid f(g_1) = h_1\}\} \\ &= \min\{f(w_1)(h_2), f(w_2)(h_1)\} \end{aligned}$$

and so

$$f(w_1)(h_1h_2h_1^{-1}) \geq \min\{f(w_1)(h_2), f(w_2)(h_1)\}.$$

Then $f(\mu_1) \preccurlyeq f(\mu_2)$. □

Proposition 4.6. *Let $\mu_1 = r_1e^{iw_1} \in CFST(H)$ and $\mu_2 = r_2e^{iw_2} \in CFST(H)$ and $f : G \to H$ be a group homomorphism. If $\mu_1 \preccurlyeq \mu_2$, then $f^{-1}(\mu_1) \preccurlyeq f^{-1}(\mu_2)$.*

Proof. Let $f^{-1}(\mu_1) = f^{-1}(r_1)e^{if^{-1}(w_1)}$ and $f^{-1}(\mu_2) = f^{-1}(r_2)e^{if^{-1}(w_2)}$ and as Proposition 4.2 we obtain that $f^{-1}(\mu_1) \in CFST(G)$ and $f^{-1}(\mu_2) \in CFST(G)$. Let

$g_1, g_2 \in G$ then

$$\begin{aligned} f^{-1}(r_1)(g_1g_2g_1^{-1}) &= r_1(f(g_1g_2g_1^{-1})) \\ &= r_1(f(g_1)f(g_2)f(g_1^{-1})) \\ &= r_1(f(g_1)f(g_2)f^{-1}(g_1)) \\ &\geq T(r_1(f(g_2)), r_2(f(g_1))) \\ &= T(f^{-1}(r_1)(g_2), f^{-1}(r_2)(g_1). \end{aligned}$$

Also

$$\begin{aligned} f^{-1}(w_1)(g_1g_2g_1^{-1}) &= w_1(f(g_1g_2g_1^{-1})) \\ &= w_1(f(g_1)f(g_2)f(g_1^{-1})) \\ &= w_1(f(g_1)f(g_2)f^{-1}(g_1)) \\ &\geq \min\{w_1(f(g_2)), w_2(f(g_1))\} \\ &= \min\{f^{-1}(w_1)(g_2), f^{-1}(w_2)(g_1\}. \end{aligned}$$

Therefore $f^{-1}(\mu_1) \preccurlyeq f^{-1}(\mu_2)$. □

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