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Fixed point theorems for cyclic contractions on intuitionistic generalized fuzzy metric spaces

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Abstract. In this work, we have brought in the notion of cyclic contraction into the intuitionistic generalized fuzzy metric spaces. We have set up some common fixed point theorems for these contractions and continuous mappings by considering their weakly commuting nature.

1. Introduction

Joseph Liouville [14] and Charles Emile Picard [16] presented independently the idea of fixed points. Stefan Banach [2] used it formally to bring out his contraction principle, the most famous in 1931 and thereon. It is here that the study of fixed points began to rise and is now densely filled with the numerous results over various settings.

One of these noteworthy settings is the intuitionistic fuzzy sets [1], an extension of fuzzy sets [21]. As the intuitionistic fuzzy setting has the features to deal with uncertainty and vagueness, a series of authors have been working on it. Several kinds of generalized spaces have been constructed. Numerous fixed point theorems, under various contractive conditions, have been established in these spaces.

In addition, in 2003, Kirk et al. introduced cyclic contractions and used these contractions to extend the most important fixed point results of Banach, Edelstein, and Caristi. Below are, respectively, these extensions.

Theorem 1.1. *(Kirk et.al. 2003)* [12] *Let A snd B be nonempty closed subsets of a complete metric space X, and suppose $F : X \to X$ satisfies (1) $F(A) \subseteq B$ and*

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$F(B) \subseteq A$, (2) $d(F(x), F(y)) \leq kd(x, y)$ for all $x \in A$ and $y \in B$, where $k \in (0, 1)$. Then F has a unique fixed point in $A \cap B$.

Theorem 1.2. *(Kirk et.al. 2003)* [12] *Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space, at least one of which is compact, and suppose $F : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ satisfies the following conditions (where $A_{p+1} = A_1$):*
(1) $F(A_i) \subseteq A_{i+1}$ for $1 \leq i \leq p$,
(2) $d(F(x), F(y)) < d(x, y)$ whenever $x \in A_i$, $y \in A_{i+1}$ and $x \neq y$, $(1 \leq i \leq p)$.
Then F has a unique fixed point.

Theorem 1.3. *(Kirk et.al. 2003)* [12] *Let $A_1, A_2, ..., A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space X, and suppose $f : X \to X$ satisfies the following conditions.*
(1) $f(A_i) \subseteq A_{i+1}$ for $1 \leq i \leq p$,
(2) $d(x, f(x)) \leq \phi_i(x) - \phi_{i+1}(f(x))$ for all $x \in A_i (1 \leq i \leq p)$, where each $\phi_i : A_i \to \mathcal{R}$ is lower semi-continuousand bounded below.
Then f has a fixed point.

In the year 2012, Karapinar et al.[11] have brought in cyclic contractions into G-metric spaces. He derived cyclic contractive version of Banach's principle as below. He also proved fixed point results for generalized cyclic weak ϕ-contractions.

Theorem 1.4. *(Karapinar et al. 2012)*[11] *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^{m}$ be a family of nonempty G-closed subsets of X. Let $Y = \cup_{j=1}^{m} A_j$ and $T : Y \to Y$ be a map satisfying $T(A_j) \subseteq A_{j+1}$, $j = 1, ..., m$, where $A_{m+1} = A_1$. If there exists $k \in (0, 1)$ such that $G(Tx, Ty, Tz) \leq kG(x, y, z)$ holds for all $x \in A_j$ and $y, z \in A_{j+1}$, $j = 1, ..., m$ then, T has a unique fixed point in $\cup_{j=1}^{m} A_j$.*

In the year 2014, Jleli and Samet [9] introduced the notion of θ-contractions. In the year 2017, Zheng et al. introduced the following notion of $\theta - \phi$ contraction and extended the fixed point results of [4, 5, 7, 8, 2, 20]

Definition 1.1. (Zheng et al. 2017) [22] Let (X, d) be a metric space and $T : X \to X$ be a mapping. For the classes of functions Φ and Θ, as defined in [20], T is said to be a $\theta - \phi$ contraction, if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$, $d(Tx, Ty) \neq 0$ implies $\theta(d(Tx, Ty)) \leq \phi[\theta(N(x, y))]$.

These findings inspired us to bring in the concept of cyclic contraction into the intuitionistic generalized fuzzy metric spaces [15] and to prove some common fixed point theorems for these contractions.

2. Preliminaries

This section begins with the definitions of triangular norms [17], the most important concepts, as they have been the main frame work in defining various metric spaces.

Definition 2.1. [17] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm, if
(i) $*$ is commutative, associative and continuous,
(ii) $t * 1 = t$, for all $t \in [0,1]$,
(iii) $t * s \leq u * v$ whenever $t \leq u$ and $s \leq v$, and $s, t, u, v \in [0,1]$.

Definition 2.2. [17] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm, if
(i) $\diamond$ is commutative, associative and continuous,
(ii) $t \diamond 0 = t$ for all $t \in [0,1]$,
(iii) $t * s \leq u * v$ whenever $t \leq u$ and $s \leq v$, and $s, t, u, v \in [0,1]$.

These norms were used by Sedghi and Shobe[18] to introduce the following $\mathfrak{M}$-fuzzy metric space as a generalisation of Dhage's D-metric[6].

Definition 2.3. [18] A 3-tuple $(Z, \mathfrak{M}, *)$ is called an $\mathfrak{M}$-fuzzy metric space, if Z is a nonempty set, $*$ is a continuous t-norm and $\mathfrak{M}$ is a fuzzy set on $Z^3 \times (0, \infty)$, satisfying the following conditions for each $\eta, \zeta, \upsilon, \xi \in Z$ and $t, s > 0$:
(i) $\mathfrak{M}(\eta, \zeta, \upsilon, t) > 0$,
(ii) $\mathfrak{M}(\eta, \zeta, \upsilon, t) = 1$ if and only if $\eta = \zeta = \upsilon$,
(iii) $\mathfrak{M}(\eta, \zeta, \upsilon, t) = \mathfrak{M}(p(\eta, \zeta, z), t)$, where p is a permutation,
(iv) $\mathfrak{M}(\eta, \zeta, \upsilon, t + s) \geq \mathfrak{M}(\eta, \zeta, \xi, t) * \mathfrak{M}(\xi, \upsilon, \upsilon, s)$,
(v) $\mathfrak{M}(\eta, \zeta, z, \cdot) : (0, \infty) \to [0,1]$ is continuous.

Following the $\mathfrak{M}$-fuzzy metric space, Muthuraj and Pandiselvi have introduced a generalization of the same in the fuzzy setting as given below.

Definition 2.4. [15] An Intuitionistic Generalized Fuzzy Metric Space (shortly, IGFM Space) is a 5-tuple $(Z, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ where Z is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and $\mathfrak{M}$ and $\mathfrak{N}$ are fuzzy sets in $Z^3 \times [0,1]$ satisfying the following conditions:
For $\eta, \zeta, z, a \in Z$ and $t, s \in [0,1]$,
(i) $\mathfrak{M}(\eta, \zeta, \upsilon, t) + \mathfrak{N}(\eta, \zeta, \upsilon, t) \leq 1$,
(ii) $\mathfrak{M}(\eta, \zeta, \upsilon, t) > 0$,
(iii) $\mathfrak{M}(\eta, \zeta, \upsilon, t) = 1$ if and only if $\eta = \zeta = \upsilon$,
(iv) $\mathfrak{M}(\eta, \zeta, \upsilon, t) = \mathfrak{M}(p(\eta, \zeta, \upsilon), t)$ where p is a permutation,
(v) $\mathfrak{M}(\eta, \zeta, \upsilon, t + s) \geq \mathfrak{M}(\eta, \zeta, \xi, t) * \mathfrak{M}(\xi, \upsilon, \upsilon, s)$,
(vi) $\mathfrak{N}(\eta, \zeta, \upsilon, \cdot) : (0, +\infty) \to [0,1]$ is continuous,
(vii) $\mathfrak{N}(\eta, \zeta, \upsilon, t) < 1$,
(viii) $\mathfrak{N}(\eta, \zeta, \upsilon, t) = 0$ if and only if $\eta = \zeta = \upsilon$,
(ix) $\mathfrak{N}(\eta, \zeta, \upsilon, t) = \mathfrak{N}(p(\eta, \zeta, \upsilon, t))$,
(x) $\mathfrak{N}(\eta, \zeta, \upsilon, t + s) \leq \mathfrak{N}(\eta, \zeta, \xi, t) \diamond \mathfrak{N}(\xi, \upsilon, \upsilon, s)$,
(xi) $\mathfrak{N}(\eta, \zeta, \upsilon, \cdot) : (0, +\infty) \to [0,1]$ is continuous.

The pair $(\mathfrak{M}, \mathfrak{N})$ is called an Intuitionistic Generalized Fuzzy Metric on Z. $\mathfrak{M}(\eta, \zeta, \upsilon, t)$ and $\mathfrak{N}(\eta, \zeta, \upsilon, t)$ give, respectively, the degree of nearness and the degree of nonnearness between η, ζ and υ with respect to t.

Definition 2.5. [13]The self-mappings $\Gamma, \Upsilon : X \to Z$ defined on an IGFM space $(X, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ are said to be weakly commuting, if for all $\eta \in Z$ and $t > 0$,

$$\mathfrak{M}(\Gamma\Upsilon\eta, \Upsilon\Gamma\eta, \Upsilon\Gamma\eta, t) \geq \mathfrak{M}(\Gamma\eta, \Upsilon\eta, \Upsilon\eta, t),$$
$$\mathfrak{M}(\Gamma\Upsilon\eta, \Upsilon\Gamma\eta, \Upsilon\Gamma\eta, t) \leq \mathfrak{N}(\Gamma\eta, \Upsilon\eta, \Upsilon\eta, t).$$

The next is the definition of cyclic representation introduced by Kirk et al.[12] which has laid the base for defining cyclic contractive conditions.

Definition 2.6. Let Z be a nonempty set and $\Gamma : Z \to Z$ be an operator $Z_1, \dots, Z_m$ be subsets of Z. Then $\bigcup_{i=1}^{m} Z_i$ is a cyclic representation of Z with respect to Γ, if
(i) $Z = \bigcup_{i=1}^{m} Z_i$, where $Z_i, i = 1, \dots, m$ are nonempty set,
(ii) $\Gamma(Z_1) \subset Z_2, \dots, \Gamma(Z_{m-1}) \subset Z_m, \Gamma(Z_m) \subset Z_1$.

Erdal Karapinar et al. [10] extend the above definition to a pair of mappings as follows:

Definition 2.7. [10] Let Z be a nonempty set, m be a positive integer and $\Gamma, \Upsilon : X \to Z$ be two mappings. $Z = \bigcup_{i=1}^{m} Z_i$ is said to be a cyclic representation of Z with respect to $(\Gamma - \Upsilon)$, if
(i) $Z_i, i = 1, \dots, m$ are nonempty sets,
(ii) $\Gamma(Z_1) \subset \Upsilon(Z_2), \dots, T(Z_{m-1}) \subset \Upsilon(Z_m), T(Z_m) \subset \Upsilon(Z_1)$.

Definition 2.8. [13]Let Z be a nonempty set. An element $\eta \in Z$ is said to be a common fixed point of the mappings $\Gamma : Z \times Z \to$ and $g : Z \to Z$, if

$$\eta = g\eta = \Gamma(\eta, \eta).$$

3. Main Results

Let us start with the following sets of functions which are essential to accomplish the intended work.
Θ is the set of all functions $\theta : [0, 1] \to [0, 1]$ satisfying the following conditions:
$[\Theta_1]$ θ is nondecreasing and continuous,
$[\Theta_2]$ $\theta(r) > 0$, if $r > 0$,
$[\Theta_3]$ $\theta(r) = r$, if $r \in \{0, 1\}$,
$[\Theta_4]$ $\theta(r) < r$, for all $r \in (0, 1)$.

Θ' is the set of all functions $\theta' : [0, 1] \to [0, 1]$ satisfying the following conditions:
$[\Theta'_1]$ θ' is nonincreasing and continuous,

$[\Theta_2']$ $\theta'(r) < 1$, if $r < 1$,
$[\Theta_3']$ $\theta'(r) = r$, if $r \in \{0, 1\}$,
$[\Theta_4']$ $\theta'(r) > r$, for all $r \in (0, 1)$.

Φ is the set of all functions $\phi : [0, 1] \to [0, 1]$ satisfying the conditions:
$[\Phi_1]$ ϕ is nondecreasing left continuous,
$[\Phi_2]$ $\phi(r) > r$, for all $r \in (0, 1)$,
$[\Phi_3]$ $\phi(1) = 1$,
$[\Phi_4]$ $\lim_{n\to\infty} \phi^n(r) = 1$, for all $r \in (0, 1)$.

Φ' is the set of all functions $\phi' : [0, 1] \to [0, 1]$ satisfying the following conditions:
$[\Phi_1']$ ϕ' is nonincreasing right continuous,
$[\Phi_2']$ $\phi'(r) < r$, for all $r \in (0, 1)$,
$[\Phi_3']$ $\phi(0) = 0$,
$[\Phi_4']$ $\lim_{n\to\infty} \phi^n(r) = 0$ for all $r \in (0, 1)$.
Rest of this section assumes that $\theta \in \Theta$, $\theta' \in \Theta'$, $\phi \in \Phi$ and $\phi' \in \Phi'$.

Definition 3.1. Let $(Z, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be an intuitionistic generalized fuzzy metric space and $\bigcup_{i=1}^{m} Z_i$ be a cyclic representation of W with respect to $\Gamma : W \to W$ where $Z_1, \ldots, Z_m$ are subsets of Z. Γ is called a cyclic $(\theta \circ \phi, \theta' \circ \phi')$-contraction, if there exists θ, θ', ϕ and ϕ' such that

$$\theta(\mathfrak{M}(T\eta, T\zeta, T\upsilon, t)) \geq \phi(\theta(\mathfrak{M}(\eta, \zeta, \upsilon, t))),$$
$$\theta'(\mathfrak{N}(T\eta, T\zeta, T\upsilon, t)) \leq \phi'(\theta'(\mathfrak{N}(\eta, \zeta, \upsilon, t))),$$

for $\eta \in Z_i$, $\zeta \in Z_{i+1}$, $\upsilon \in Z_{i+2}$, $i = 1, \ldots, m$.

Theorem 3.1. *Let Z_1 and Z_2 be a closed subset of Z and $W = Z_1 \cup Z_2$ and $F : W \times W \to W$ and $G : W \to W$ be such that*

$$(i) F \text{ is a cyclic } (\phi \circ \theta, \phi' \circ \theta')\text{−contraction}, \tag{1}$$
$$(ii) F(Z_1) \subset G(Z_2), F(Z_2) \subset G(Z_1). \tag{2}$$

Then F has a unique fixed point.

Proof. Let $\eta_0 \in Z_1$ and $\xi_0 \in Z_2$.
Consider the sequences $\{\eta_n\}$ and $\{\xi_n\}$ where

$$\eta_{n+1} = F(\eta_n, \xi_n), \qquad \xi_{n+1} = F(\xi_n, \eta_n).$$

From (1),

$$\begin{aligned}\mathfrak{M}(\eta_n, \eta_n, \eta_{n+1}, t) &\geq \theta\big(\mathfrak{M}(\eta_n, \eta_n, \eta_{n+1}, t)\big)\\ &= \theta\big(\mathfrak{M}(F(\eta_{n-1}, \xi_{n-1}), F(\eta_{n-1}, \xi_{n-1}), t)\big)\\ &\geq \phi\Big(\theta\big(\mathfrak{M}(\eta_{n-1}, \eta_{n-1}, \eta_n, t)\big)\Big)\end{aligned}$$

$$\geq \phi^n\Big(\theta\big(\mathfrak{M}(\eta_0,\eta_0,\eta_1,t)\big)\Big).$$

For $m>0$,

$$\mathfrak{M}(\eta_n,\eta_n,\eta_{n+m},t)\geq \phi^n\Big(\theta\big(\mathfrak{M}(\eta_0,\eta_0,\eta_1,\frac{t}{m})*\cdots*\phi^{k+m-1}\theta\big(\mathfrak{M}(\eta_0,\eta_0,\eta_1,\frac{t}{m}).\quad (3)$$

From (2),

$$\begin{aligned}\mathfrak{N}(\eta_n,\eta_n,\eta_{n+1},t)&\leq \theta'\big(\mathfrak{N}(\eta_n,\eta_n,\eta_{n+1},t)\\&=\theta'\big(\mathfrak{N}(F(\eta_{n-1},\xi_{n-1}),F(\eta_{n-1},\xi_{n-1}),t)\big)\\&\leq \phi'\Big(\theta'\big(\mathfrak{N}(\eta_{n-1},\eta_{n-1},\eta_n,t)\big)\Big).\end{aligned}$$

For $m>0$,

$$\mathfrak{N}(\eta_n,\eta_n,\eta_{n+m},t)\leq \phi'^n\Big(\theta'\big(\mathfrak{N}(\eta_0,\eta_0,\eta_1,\frac{t}{m})*\cdots*\phi'^{k+m-1}\theta'\big(\mathfrak{N}(\eta_0,\eta_0,\eta_1,\frac{t}{m}).\quad (4)$$

Letting $n\to\infty$ in (3) and (4), the properties of ϕ and ϕ' make the sequence $\{\eta_n\}$ Cauchy.
From (1),

$$\begin{aligned}\theta\big(\mathfrak{M}(\xi_n,\xi_n,\xi_{n+1},t)&=\phi\mathfrak{M}\big(F(\xi_{n-1},\zeta_{n-1}),F(\xi_{n-1},\zeta_{n-1}),F(\xi_n,\zeta_n),t\big)\\&\geq \phi\theta\big(\mathfrak{M}\big(\xi_{n-1},\xi_{n-1},t\big).\end{aligned}$$

This gives,

$$\mathfrak{M}(\xi_n,\xi_n,\xi_{n+1},t)\geq \mathfrak{M}\left(\xi_n,\xi_n,\xi_{n+1},\frac{t}{m}\right)*\cdots*\mathfrak{M}\left(\xi_{n+m-1},\xi_{n+m-1},\xi_{n+m},\frac{t}{m}\right).$$

This gives,

$$\theta\big(\mathfrak{M}(\xi_n,\xi_n,\xi_{n+1},t)\geq \phi^n\theta\big(\mathfrak{M}\left(\xi_0,\xi_0,\xi_1,\frac{t}{m}\right)\big)*\cdots*\phi^{n+m-1}\Big(\theta\big(\mathfrak{M}\left(\xi_0,\xi_0,\xi_1,\frac{t}{m}\right)\big)\Big).\quad (5)$$

From (2),

$$\begin{aligned}\theta'\big(\mathfrak{N}(\xi_n,\xi_n,\xi_{n+1},t)\big)&=\phi'\mathfrak{N}\big(F(\xi_{n-1},\zeta_{n-1}),F(\xi_{n-1},\zeta_{n-1}),F(\xi_n,\zeta_n),t\big)\\&\leq \phi'\Big(\theta'\big(\mathfrak{N}\big(\xi_{n-1},\xi_{n-1},t\big)\big)\Big).\end{aligned}$$

This gives,

$$\mathfrak{N}(\xi_n,\xi_n,\xi_{n+1},t)\leq \mathfrak{N}\left(\xi_n,\xi_n,\xi_{n+1},\frac{t}{m}\right)*\cdots*\mathfrak{N}\left(\xi_{n+m-1},\xi_{n+m-1},\xi_{n+m},\frac{t}{m}\right).$$

This gives,

$$\theta'\big(\mathfrak{N}(\xi_n,\xi_n,\xi_{n+1},t)\leq \phi'^n\theta'\big(\mathfrak{N}\left(\xi_0,\xi_0,\xi_1,\frac{t}{m}\right)*\cdots*\phi'^{n+m-1}\theta'\big(\mathfrak{N}\left(\xi_0,\xi_0,\xi_1,\frac{t}{m}\right).\quad (6)$$

Taking $n \to \infty$ in (5) and (6), the sequence $\{\xi_n\}$ becomes Cauchy.
Therefore, we can find $\upsilon \in Z_1$ and $\xi \in Z_2$ such that $\upsilon_n \to \upsilon, \xi_n \to \xi$. As all the sub sequences of $\{\upsilon_n\}$ and $\{\xi_n\}$ converge, respectively, to υ and ξ, we have that $\upsilon, \xi \in W$.
Let $F/_W : W \times W \to W$ be the restricted function of F to W.
Then from (1) and (2),

$$\mathfrak{M}\big(F/_W(\upsilon,\xi), F/_W(\upsilon,\xi), \upsilon, t\big) \geq \phi\Big(\theta\big(\mathfrak{M}\left(\upsilon,\upsilon,\upsilon_{n+1},\frac{t}{\theta}\right) * \mathfrak{M}\left(\upsilon,\upsilon,\upsilon_{n+1},\frac{t}{\theta}\right)\big)\Big),$$

$$\mathfrak{N}\big(F/_W(\upsilon,\xi), F/_W(\upsilon,\xi), t\big) \leq \phi'\Big(\theta'\big(\mathfrak{N}\left(\upsilon,\upsilon,\upsilon_{n+1},\frac{t}{\theta'}\right) * \mathfrak{N}\left(\upsilon,\upsilon,\upsilon_{n+1},\frac{t}{\theta'}\right)\big)\Big),$$

which gives, $F/_W(\upsilon,\xi) = \upsilon \in W$.

$$\mathfrak{M}\big(F/_W(\xi,\upsilon), F/_W(\xi,\upsilon), \xi, t\big) \geq \phi\Big(\theta\big(\mathfrak{M}\left(\xi,\xi,\xi_{n+1},\frac{t}{2}\right)\big)\Big) * \phi\Big(\theta\big(\mathfrak{M}\left(\xi,\xi,\xi_{n+1},\frac{t}{2}\right)\big)\Big),$$

$$\mathfrak{N}\big(F/_W(\xi,\upsilon), F/_W(\xi,\upsilon), \xi, t\big) \leq \phi'\Big(\theta'\big(\mathfrak{N}\left(\xi,\xi,\xi_{n+1},\frac{t}{2}\right)\big)\Big) * \phi'\Big(\theta'\big(\mathfrak{N}\left(\xi,\xi,\xi_{n+1},\frac{t}{2}\right)\big)\Big),$$

which gives, $F/_W(\xi,\upsilon) = \xi \in W$.
Now,

$$\theta\big(\mathfrak{M}(\upsilon,\upsilon,\xi,t)\big) \geq \phi\Big(\theta\big(\mathfrak{M}(\upsilon,\upsilon,\xi,t)\big)\Big) \geq \theta\big(\mathfrak{M}(\upsilon,\upsilon,\xi,t)\big),$$

$$\theta'\big(\mathfrak{N}(\upsilon,\upsilon,\xi,t)\big) \leq \phi'\Big(\theta'\big(\mathfrak{N}(\upsilon,\upsilon,\xi,t) \leq \theta'\big(\mathfrak{N}(\upsilon,\upsilon,\xi,t)\big)\Big).$$

Therefore, $\upsilon = \xi$ and hence we have $F(\upsilon,\upsilon) = \upsilon$. □

Theorem 3.2. *Let $(Z,\mathfrak{M},\mathfrak{N},*,\diamond)$ be a complete IGFM space. Let Z_1 and Z_2 be closed subsets of Z and $W = Z_1 \cup Z_2$. Consider $F : W \times W \to W$ and continuous functions $G, H : W \to W$ satisfying the following conditions:*
(i) $F(W \times W) \subset G(W) \cap H(W)$,
(ii) F is a cyclic $(\theta \circ \phi, \theta' \circ \phi')$-contraction,
(iii)

$$\theta\big(\mathfrak{M}(F(\eta,\zeta),F(\eta,\zeta),F(\xi,\upsilon),c)\big) \geq \phi\left(\theta\left(\min\left\{\begin{array}{c}\mathfrak{M}(G\eta,G\eta,H\xi,c),\\ \mathfrak{M}(G\eta,G\eta,F(\eta,\zeta),c),\\ \mathfrak{M}(G\eta,G\eta,F(\xi,\upsilon),c),\\ \mathfrak{M}(H\xi,H\xi,F(\xi,\upsilon),c)\end{array}\right\}\right)\right), \tag{7}$$

$$\theta'\big(\mathfrak{N}(F(\eta,\zeta),F(\eta,\zeta),F(\xi,\upsilon),c)\big) \leq \phi'\left(\theta'\left(\max\left\{\begin{array}{c}\mathfrak{N}(G\eta,G\eta,H\xi,c),\\ \mathfrak{N}(G\eta,G\eta,F(\eta,\zeta),c),\\ \mathfrak{N}(G\eta,G\eta,F(\xi,\upsilon),c),\\ \mathfrak{N}(H\xi,H\xi,F(\xi,\upsilon),c)\end{array}\right\}\right)\right), \tag{8}$$

for $\eta, \zeta \in Z_1,\ \xi, \upsilon \in Z_2$.
(iv) (F, G) *and* (F, H) *are weakly commuting pairs.*
Then F, G *and* H *have a unique common fixed point in* W.

Proof. Let $\eta_0 \in Z_1$ and $\zeta_0 \in Z_2$.
Consider $\eta_n, \upsilon_n \in Z_1$ and $\zeta_n, \xi_n \in Z_2$, where

$$\begin{aligned} F(\eta_n, \zeta_n) &= G\eta_{n+1} = \upsilon_n, \\ F(\zeta_n, \eta_n) &= G\zeta_{n+1} = \xi_n, \\ F(\eta_{n+1}, \zeta_{n+1}) &= H\eta_{n+2} = \upsilon_{n+1}, \\ F(\zeta_{n+1}, \eta_{n+1}) &= H\zeta_{n+2} = \xi_{n+1}. \end{aligned}$$

From (12),

$$\begin{aligned} \theta\Big(\mathfrak{M}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t)\Big) &= \theta\big(F(\eta_{n+2}, \zeta_{n+2}), F(\eta_{n+2}, \zeta_{n+2}), F(\eta_{n+1}, \zeta_{n+1}), t\big) \\ &\geq \phi\left(\theta\left(\min\left\{\begin{array}{c} \mathfrak{M}(G\eta_{n+2}, G\eta_{n+2}, H\eta_{n+1}, t), \\ \mathfrak{M}(G\eta_{n+2}, G\eta_{n+2}, F(\eta_{n+2}, \zeta_{n+2}), t), \\ \mathfrak{M}(G\eta_{n+2}, G\eta_{n+2}, F(\eta_{n+1}, \zeta_{n+1}), t), \\ \mathfrak{M}(H\eta_{n+1}, H\eta_{n+1}, F(\eta_{n+1}, \zeta_{n+1}), t) \end{array}\right\}\right)\right) \\ &= \phi\left(\theta\left(\min\left\{\begin{array}{c} \mathfrak{M}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_n, t), \\ \mathfrak{M}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_{n+2}, t), \\ \mathfrak{M}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_{n+1}, t), \\ \mathfrak{M}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t) \end{array}\right\}\right)\right) \\ &= \phi\left(\theta\left(\min\left\{\begin{array}{c} \mathfrak{M}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_n, t), \\ \mathfrak{M}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t) \end{array}\right\}\right)\right). \end{aligned}$$

Since $\phi(t) > t$, we get,

$$\theta\big(\mathfrak{M}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t)\big) \geq \phi\big(\theta(\mathfrak{M}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_n, c))\big). \tag{9}$$

Repeated application of (9) gives

$$\theta\big(\mathfrak{M}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t)\big) > \phi^n\big(\theta(\mathfrak{M}(\upsilon_1, \upsilon_1, \upsilon_0))\big).$$

Since $\lim_{n\to\infty} \phi^n(t) = 1$, we have

$$\lim_{n\to\infty} \theta\big(\mathfrak{M}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t)\big) = 1. \tag{10}$$

From (13),

$$\begin{aligned} \theta'\Big(\mathfrak{N}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t)\Big) &= \theta'\big(F(\eta_{n+2}, \zeta_{n+2}), F(\eta_{n+2}, \zeta_{n+2}), F(\eta_{n+1}, \zeta_{n+1}), t\big) \\ &\leq \phi'\left(\theta'\left(\min\left\{\begin{array}{c} \mathfrak{N}(G\eta_{n+2}, G\eta_{n+2}, H\eta_{n+1}, t), \\ \mathfrak{N}(G\eta_{n+2}, G\eta_{n+2}, F(\eta_{n+2}, \zeta_{n+2}), t), \\ \mathfrak{N}(G\eta_{n+2}, G\eta_{n+2}, F(\eta_{n+1}, \zeta_{n+1}), t), \\ \mathfrak{N}(H\eta_{n+1}, H\eta_{n+1}, F(\eta_{n+1}, \zeta_{n+1}), t) \end{array}\right\}\right)\right) \end{aligned}$$

$$= \phi' \left(\theta' \left(\min \left\{ \begin{array}{c} \mathfrak{N}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_n, t), \\ \mathfrak{N}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_{n+2}, t), \\ \mathfrak{N}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_{n+1}, t), \\ \mathfrak{N}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t) \end{array} \right\} \right) \right)$$

$$= \phi' \left(\theta' \left(\min \left\{ \begin{array}{c} \mathfrak{N}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_n, t), \\ \mathfrak{N}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t) \end{array} \right\} \right) \right).$$

Since $\phi'(t) < t$, we get,

$$\theta'\big(\mathfrak{N}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t)\big) \le \phi'\big(\theta'(\mathfrak{N}(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_n, c))\big). \tag{11}$$

Repeated application of (11) gives

$$\theta'\big(\mathfrak{N}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t)\big) < \phi'^n\big(\theta'(\mathfrak{N}(\upsilon_1, \upsilon_1, \upsilon_0))\big).$$

Since $\lim_{n\to\infty} \phi'^n(t) = 0$, we have

$$\lim_{n\to\infty} \theta\big(\mathfrak{N}(\upsilon_{n+2}, \upsilon_{n+2}, \upsilon_{n+1}, t)\big) = 0. \tag{12}$$

(10) and (12) make the sequences $\{\upsilon_n\}$, and hence $\{\xi_n\}$, Cauchy. Therefore we can find $\upsilon \in Z_1$ and $\xi \in Z_2$ such that $\upsilon_n \to \upsilon$ and $\xi_n \to \xi$. This makes $\{F(\eta_n, \zeta_n)\} \to \upsilon$ and $\{F(\zeta_n, \eta_n)\} \to \xi$.
Now,

$$\mathfrak{M}(\upsilon_n, \upsilon_n, F(\upsilon, \xi), t) \ge \theta\big(\mathfrak{M}(\upsilon_n, \upsilon_n, F(\upsilon, \xi), t)\big) \ge \phi\big(\theta(\mathfrak{M}(\eta_n, \eta_n, \upsilon, t))\big),$$
$$\mathfrak{N}(\upsilon_n, \upsilon_n, F(\upsilon, \xi), t) \le \theta'\big(\mathfrak{N}(\upsilon_n, \upsilon_n, F(\upsilon, \xi), t)\big) \le \phi'\big(\theta'(\mathfrak{N}(\eta_n, \eta_n, \upsilon, t))\big).$$

Letting $n \to \infty$, we get,

$$\mathfrak{M}(\upsilon, \upsilon, F(\upsilon, \xi), t) \ge \phi\big(\theta(\mathfrak{M}(\eta_n, \eta_n, \upsilon, t))\big),\ \mathfrak{N}(\upsilon, \upsilon, F(\upsilon, \xi), t) \le \phi'\big(\theta'(\mathcal{N}(\eta_n, \eta_n, \upsilon, t))\big)$$

This gives $\mathfrak{M}(\upsilon, \upsilon, F(\upsilon, \xi), t) = 1$ and $\mathfrak{N}(\upsilon, \upsilon, F(\upsilon, \xi), t) = 0$. Therefore, $F(\upsilon, \xi) = \upsilon$. In a similar way, we can obtain that $F(\upsilon, \xi) = \xi$. Since all the subsequences of $\{F(\eta_n, \zeta_n)\}$ converge to υ, and, all the subsequences of $\{F(\zeta_n, \eta_n)\}$ converge to ξ, we have $G\eta_n \to \upsilon$, $H\eta_n \to \upsilon$, $G\zeta_n \to \xi$, and $H\zeta_n \to \xi$. Hence, we have

$$G(F(\eta_n, \zeta_n)) \to G\upsilon, \qquad G(F(\zeta_n, \eta_n)) \to G\xi,$$
$$G^2\eta_n \to G\upsilon, \qquad G^2\zeta_n \to G\xi.$$

Since (F, H) and (F, G) are weakly commuting, we have

$$F(H\eta_n, H\zeta_n) \to H\upsilon,\ F(H\zeta_n, H\eta_n) \to H\xi, F(G\eta_n, G\zeta_n) \to G\upsilon,\ F(G\zeta_n, G\eta_n) \to G\xi.$$

Then from (12) and (12), we get

$$H\upsilon = \upsilon, H\xi = \xi, G\upsilon = \upsilon, G\xi = \xi.$$

Hence, we have $F(\upsilon, \xi) = H\upsilon = G\upsilon = \upsilon$, $F(\xi, \upsilon) = H\xi = G\xi = \xi$. If we take $\upsilon \ne \xi$, then from (12) and (13), we will get the following contradictions:

$$\begin{aligned}\theta\big(\mathfrak{M}(\upsilon,\upsilon,\xi,t)\big) &= \theta\big(\mathfrak{M}(F(\upsilon,\xi),F(\upsilon,\xi),F(\xi,\upsilon),t)\big)\\ &\geq \phi\big(\theta(\mathfrak{M}(\upsilon,\upsilon,\xi,t))\big)\\ &> \theta\big(\mathfrak{M}(\upsilon,\upsilon,\xi,t)\big),\end{aligned}$$

$$\begin{aligned}\theta'\big(\mathfrak{N}(\upsilon,\upsilon,w,t)\big) &= \theta'\big(\mathfrak{N}(F(\upsilon,\xi),F(\upsilon,\xi),F(\xi,\upsilon),t)\big)\\ &\leq \phi'\big(\theta'(\mathfrak{N}(\upsilon,\upsilon,\xi,t))\big)\\ &< \theta'\big(\mathfrak{N}(\upsilon,\upsilon,\xi,t)\big).\end{aligned}$$

Therefore, $\upsilon=\xi$ and $F(\upsilon,\upsilon)=G\upsilon=H\upsilon=\upsilon$. Since Z_1 and Z_2 intersect, we have that $\upsilon\in Z_1\cap Z_2=W$. □

Corollary 3.2. Let $(Z,\mathfrak{M},\mathfrak{N},*,\diamond)$ be a complete IGFM space. Let Z_1 and Z_2 be closed subsets of Z and $W=Z_1\cup Z_2$. Consider $F:W\times W\to W$ and a continuous function $G:W\to W$ satisfying the following conditions:
(i) $F(W\times W)\subset G(W)$,
(ii) F is a cyclic $(\theta\circ\phi,\theta'\circ\phi')$-contraction,
(iii)

$$\theta\big(\mathfrak{M}(F(\eta,\zeta),F(\eta,\zeta),F(\xi,\upsilon),c)\big)\geq\phi\left(\theta\left(\min\left\{\begin{array}{c}\mathfrak{M}(G\eta,G\eta,G\xi,c),\\ \mathfrak{M}(G\eta,G\eta,F(\eta,\zeta),c),\\ \mathfrak{M}(G\eta,G\eta,F(\xi,\upsilon),c),\\ \mathfrak{M}(G\xi,G\xi,F(\xi,\upsilon),c)\end{array}\right\}\right)\right), \tag{12}$$

$$\theta'\big(\mathfrak{N}(F(\eta,\zeta),F(\eta,\zeta),F(\xi,\upsilon),c)\big)\leq\phi'\left(\theta'\left(\max\left\{\begin{array}{c}\mathfrak{N}(G\eta,G\eta,G\xi,c),\\ \mathfrak{N}(G\eta,G\eta,F(\eta,\zeta),c),\\ \mathfrak{N}(G\eta,G\eta,F(\xi,\upsilon),c),\\ \mathfrak{N}(G\xi,G\xi,F(\xi,\upsilon),c)\end{array}\right\}\right)\right), \tag{13}$$

for $\eta,\xi\in Z_1,\zeta,\upsilon\in Z_2$.
(iv) (F,G) is weakly commuting. Then F and G have a unique common fixed point.

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