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Quasi fixed points of contractions on semi cone metric spaces

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Abstract. In this article, the concept of semi-metric spaces is extended to semi-cone metric spaces. Also, some theorems for quasi-fixed points of Banach contractions, Kannan's contractions, and Chatterjea's contractions on semi-cone metric spaces are proved. Moreover, some quasi-fixed point results for different types of contraction mappings on semi-cone metric spaces are derived.

1. Introduction

S. I. Raj extended the concept of fixed points in metric spaces to quasi fixed points in semi metric spaces in Chapter 5 of [10]. Also, he established a quasi fixed point result for Banach contractions on semi metric spaces in it.

In [4], L. G. Huang and X. Zhang invented the notion of cone metric space(or, CMS). Every metric space is a CMS with respect to the natural cone $P = [0, \infty)$ of the real line Banach space $\mathbb{R}$. The reason for the introduction of CMS for fixed point theory is the existence of some mappings on the Euclidean plane into itself which are not contractive with respect to the usual Euclidean metric, but contractive with respect to cone metric; for instance, see the final example provided in the article [4]. So, many articles for metric fixed point theory were extended from metric spaces to cone metric spaces(or, CMSs), for example, see [1, 5, 11, 12, 14]. Recently many articles are being appeared for fixed point theory in CMSs, for example, see [3, 7, 8, 13, 15].

In this article, we generalize the extended Banach contraction theorem for quasi fixed points(or, QFPs) in semi metric spaces in [10] for Banach contraction in semi

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cone metric spaces(SCMSs). Also, we prove some more theorems for Kannan's contraction [6], Chatterjea's contraction [2], and different types of contraction mappings in order complete SCMSs. Recently fixed point theorems derived for generalized Kannan's contraction, and generalized Chatterjea's contraction in [9].

2. Cone metric spaces

In this section, we generalize some definitions and known results of [4] to SCMSs. Let E always be a real Banach space and P a subset of E. The set P is called a cone if and only if (or, iff)

(I) P is closed, $P \neq \emptyset$, and $P \neq \{0\}$;
(II) $x, y \in R$, $x, y \geq 0$, $\psi, \tau \in P \Rightarrow x\psi + y\tau \in P$; and
(III) $\psi \in P$ and $-\psi \in P \Rightarrow \psi = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $\psi \leq \tau$ iff $\tau - \psi \in P$. We shall write $\psi < \tau$ to indicate that $\psi \leq \tau$ but $\psi \neq \tau$, and $\psi \ll \tau$ iff $\tau - \psi \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $\psi, \tau \in E$ satisfying $0 \leq \psi \leq \tau$ we have $\| \psi \| \leq K \| \tau \|$. The least positive number K is called the normal constant of P.

In the following, E is a Banach space, P is a cone in E with $intP \neq \emptyset$, and $\leq$ is the partial ordering induced by P.

Definition 2.1. [4] Let $S \neq \emptyset$ be a set. The mapping $d : S \times S \to E$ is called a cone metric on S if

(i) $0 \leq d(\psi, \tau), \forall \psi, \tau \in S$ and
(ii) $d(\psi, \tau) = 0$ iff $\psi = \tau$; and
(iii) $d(\psi, \tau) = d(\tau, \psi)$, $\forall \psi, \tau \in S$; and
(iv) $d(\psi, \tau) \leq d(\psi, \nu) + d(\nu, \tau), \forall \psi, \nu, \tau \in S$.

Then (S, d) is called a cone metric space(CMS).

Definition 2.2. Let $S \neq \emptyset$ be a set. The mapping $d : S \times S \to E$ is said to be a semi cone metric on S if (i), (iii) and (iv) of Definition 2.1 are satisfied, and $\psi = \tau$ implies $d(\psi, \tau) = 0$ but $d(\psi, \tau) = 0$ need not imply $\psi = \tau$. Then (S, d) is called a semi cone metric space(or, SCMS).

Definition 2.3. Let (S, d) be a SCMS. Let $\{\psi_n\}$ be a sequence in S. Then

(I) $\{\psi_n\}$ is called an order convergent to ψ if for every $c \in E$ with $0 \ll c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(\psi_n, \psi) \ll c$, for some $\psi \in S$.
(II) $\{\psi_n\}$ is said to be order Cauchy sequence(or, OCS) if for any $c \in E$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n, m > N$, $d(\psi_n, \psi_m) \ll c$.
(III) (S, d) is called order complete, if every OCS is order convergent in S.

Lemma 2.1. *Let (S,d) be a SCMS, and P be a normal cone. Let $\{\psi_n\}$ be a sequence in S. Then*

(I) *$\{\psi_n\}$ order converges to ψ in S iff $d(\psi_n,\psi)\to 0$ as $n\to\infty$.*
(II) *$\{\psi_n\}$ is an OCS iff $d(\psi_n,\psi_m)\to 0$ as $n,m\to\infty$.*

Proof. It is alike to the proof of Lemma 1 and Lemma 4 of [4]. □

Since Lemma 2.1, we get the next Remark 2.4.

Remark 2.4. Suppose that for every sequence $\{\psi_n\}$ in S such that $d(\psi_n,\psi_m)\to 0$ as $n,m\to\infty$, there is a point ψ in S such that $d(\psi_n,\psi)\to 0$ as $n\to\infty$. Then (S,d) is an order complete SCMS. The converse is also true.

Definition 2.5. Let (S,d) be a SCMS and f be a function on S to itself. A point $\psi\in S$ is quasi fixed point(or, QFP) if $d(f(\psi),\psi)=0$. Uniqueness of the QFP ψ means that $d(\psi,\tau)=0$ whenever τ is also a QFP.

3. fixed point theorems

The next one is a generalization of main theorem of [10] for Banach contraction, to SCMSs.

Theorem 3.1. *Let (S,d) be an order complete SCMS, and P be a normal cone with normal constant(or, NCWNC) K. Let f be a given function on S to itself such that $d(f(\psi),f(\tau))\leq qd(\psi,\tau)$, $\forall\psi,\tau\in S$, satisfying $d(\psi,\tau)\neq 0$, for some $q\in(0,1)$. Then f has a unique QFP ψ^* in S. If $d(\tau,\psi^*)=0=d(f(\tau),f(\psi^*))$, then τ is also a QFP of f.*

Proof. Define $\psi_{n+1}=f(\psi_n)$, where $n=0,1,2,3,...$, and ψ_0 is fixed in S. If $d(\psi_{n+1},\psi_n)=0$, for some $n=0,1,2,3,...$, then ψ_n is a QFP. Suppose $d(\psi_{n+1},\psi_n)\neq 0$, for every $n=0,1,2,3,....$ Then we have

$$\begin{aligned} d(\psi_{n+1},\psi_n)=d(f(\psi_n),f(\psi_{n-1})) &\leq qd(\psi_n,\psi_{n-1})\\ &\leq q^2d(\psi_{n-1},\psi_{n-2})\leq ...\leq q^nd(\psi_1,\psi_0). \end{aligned}$$

For $m>n\geq 1$, we have

$$\begin{aligned} d(\psi_m,\psi_n) &\leq d(\psi_m,\psi_{m-1})+d(\psi_{m-1},\psi_{m-2})+...+d(\psi_{n+1},\psi_n)\\ &\leq (q^{m-1}+q^{m-2}+...+q^n)d(\psi_1,\psi_0)\\ &\leq \Big(\frac{q^n}{1-q}\Big)d(\psi_1,\psi_0). \end{aligned}$$

We get $\|d(\psi_m,\psi_n)\|\leq(\frac{q^n}{1-q})K\|d(\psi_1,\psi_0)\|$. This implies that $d(\psi_n,\psi_m)\to 0$ as $n,m\to\infty$. By Remark 2.4, there is a $\psi^*\in S$ such that $d(\psi_n,\psi^*)\to 0$ as $n\to\infty$.

Also, when $d(\psi_n, \psi^*) \neq 0$, we have

$$\begin{aligned} d(f(\psi^*), \psi^*) &\leq d(f(\psi^*), f(\psi_n)) + d(f(\psi_n), \psi^*) \\ &\leq qd(\psi^*, \psi_n) + d(\psi_{n+1}, \psi^*) \\ ||d(f(\psi^*), \psi^*)|| &\leq qK||d(\psi^*, \psi_n)|| + K||d(\psi_{n+1}, \psi^*)||. \end{aligned}$$

If $d(\psi_n, \psi^*) \neq 0$, for infinitely many n, then $d(\psi^*, f(\psi^*)) = 0$. If $d(\psi_n, \psi^*) = 0$ except for finitely many n, then

$$\begin{aligned} d(\psi_n, \psi_{n+1}) &\leq d(\psi_n, \psi^*) + d(\psi^*, \psi_{n+1}) \\ ||d(\psi_n, \psi_{n+1})|| &\leq K||d(\psi_n, \psi^*)|| + K||d(\psi^*, \psi_{n+1})||. \end{aligned}$$

So $d(\psi_n, \psi_{n+1}) = 0$, for atleast one n. This is a contradiction, because $d(\psi_n, \psi_{n+1}) \neq 0$, $\forall n$. Therefore $d(f(\psi^*), \psi^*) = 0$, and ψ^* is a QFP.

If f has two quasi fixed points(or, QFPs) ψ and τ, then $d(\psi, \tau) = 0$ or $d(\psi, \tau) \neq 0$ and in this case we have

$$\begin{aligned} d(\tau, \psi) &\leq d(\tau, f(\tau)) + d(f(\tau), f(\psi)) + d(f(\psi), \psi) \\ &= d(f(\tau), f(\psi)) \\ &\leq qd(\tau, \psi) \\ d(\tau, \psi) &\leq q^m d(\tau, \psi), \text{ for every } m = 1, 2, 3, ..., \text{ and} \\ ||d(\tau, \psi)|| &\leq q^m K||d(\tau, \psi)||, \text{ for every } m = 1, 2, 3, ... \end{aligned}$$

Since $(q)^m K \to 0$ as $m \to \infty$, we get $d(\psi, \tau) = 0$, which is a contradiction and f has a unique QFP. If ψ^* is a QFP and $d(\tau, \psi^*) = 0 = d(f(\tau), f(\psi^*))$, for some $\tau \in S$, then

$$d(f(\tau), \tau) \leq d(f(\tau), f(\psi^*)) + d(f(\psi^*), \psi^*) + d(\psi^*, \tau) = 0$$

Therefore, $d(f(\tau), \tau) = 0$, and τ is also a QFP. □

Example 3.1. Let $P = \{(b, z) \in E : b, z \geq 0\}$ be a subset of the real Banach space $E = R^2$, the Euclidean plane. Then P is a NCWNC $K = 1$. Let $S = R^2$, and $d : S \times S \to E$ be defined by $d(\psi, \tau) = d((\psi_1, \psi_2), (\tau_1, \tau_2)) = \Big(|\psi_1 - \tau_1|, \beta|\psi_1 - \tau_1|\Big)$, $\forall \psi = (\psi_1, \psi_2), \tau = (\tau_1, \tau_2) \in S$, where $\beta \geq 1$ is a fixed constant. Then (S, d) is an order complete SCMS with respect to P. Define $T : S \to S$ by $T(\psi) = T(\psi_1, \psi_2) = (\frac{\psi_1}{3}, \frac{\psi_2}{3})$. For $\psi = (\psi_1, \psi_2), \tau = (\tau_1, \tau_2) \in S$, we have $d(T(\psi), T(\tau)) \leq \frac{1}{3} d(\psi, \tau)$. Then, hypotheses of Theorem 3.1 are satisfied with $q = \frac{1}{3}$. Also, the unique QFP is zero.

Remark 3.2. Let (S, d) be an order complete SCMS, and P be a NCWNC K. Let f be a given function on S to itself. If ψ^* is a QFP of f and $d(\tau, \psi^*) = 0 = d(f(\tau), f(\psi^*))$, then τ is also a QFP of f.

Remark 3.2 is true in the following all theorems.
The next theorem consider Kannan's contractions [6], for QFPs in SCMSs.

Theorem 3.2. *Let (S,d) be an order complete SCMS, and P be a NCWNC K. Let f be a given function on S to itself such that $d(f(\psi),f(\tau)) \leq q[d(f(\psi),\psi) + d(f(\tau),\tau)]$, $\forall \psi,\tau \in S$, satisfying $d(\psi,\tau) \neq 0$, for some $q \in (0,\frac{1}{2})$. Then f has a unique QFP ψ^* in S.*

Proof. Define $\psi_{n+1} = f(\psi_n)$, where $n = 0,1,2,3,...$, and ψ_0 is fixed in S. If $d(\psi_{n+1},\psi_n) = 0$, for some $n = 0,1,2,3,...$, then ψ_n is a QFP. Suppose $d(\psi_{n+1},\psi_n) \neq 0$, for every $n = 0,1,2,3,....$ Then, with $l = (\frac{q}{1-q})$ we have

$$\begin{aligned} d(\psi_{n+1},\psi_n) = d(f(\psi_n),f(\psi_{n-1})) &\leq q[d(f(\psi_n),\psi_n) + d(f(\psi_{n-1}),\psi_{n-1})] \\ &= q[d(\psi_{n+1},\psi_n) + d(\psi_n,\psi_{n-1})] \\ d(\psi_{n+1},\psi_n) &\leq \left(\frac{q}{1-q}\right) d(\psi_n,\psi_{n-1}) \\ &= l d(\psi_n,\psi_{n-1}) \\ &\leq l^2 d(\psi_{n-1},\psi_{n-2}) \leq ... \leq l^n d(\psi_1,\psi_0). \end{aligned}$$

For $m > n \geq 1$, we have

$$\begin{aligned} d(\psi_m,\psi_n) &\leq d(\psi_m,\psi_{m-1}) + d(\psi_{m-1},\psi_{m-2}) + ... + d(\psi_{n+1},\psi_n) \\ &\leq (l^{m-1} + l^{m-2} + ... + l^n) d(\psi_1,\psi_0) \\ &\leq \left(\frac{l^n}{1-l}\right) d(\psi_1,\psi_0). \end{aligned}$$

We get $\|d(\psi_m,\psi_n)\| \leq (\frac{l^n}{1-l})K\|d(\psi_1,\psi_0)\|$. This implies that $d(\psi_n,\psi_m) \to 0$ as $n,m \to \infty$. By Remark 2.4, there is a $\psi^* \in S$ such that $d(\psi_n,\psi^*) \to 0$ as $n \to \infty$. Also, when $d(\psi_n,\psi^*) \neq 0$, we have

$$\begin{aligned} d(f(\psi^*),\psi^*) &\leq d(f(\psi^*),f(\psi_n)) + d(f(\psi_n),\psi^*) \\ &\leq q[d(f(\psi^*),\psi^*) + d(f(\psi_n),\psi_n)] + d(\psi_{n+1},\psi^*) \\ d(f(\psi^*),\psi^*) &\leq l[d(\psi_{n+1},\psi_n)] + \left(\frac{1}{1-q}\right) d(\psi_{n+1},\psi^*) \\ &\leq l[d(\psi_{n+1},\psi^*) + d(\psi^*,\psi_n)] + \left(\frac{1}{1-q}\right) d(\psi_{n+1},\psi^*), \text{ and} \\ ||d(f(\psi^*),\psi^*)|| &\leq K||l[d(\psi_{n+1},\psi^*) + d(\psi^*,\psi_n)] + \left(\frac{1}{1-q}\right) d(\psi_{n+1},\psi^*)||. \end{aligned}$$

If $d(\psi_n,\psi^*) \neq 0$, for infinitely many n, then $d(\psi^*,f(\psi^*)) = 0$. If $d(\psi_n,\psi^*) = 0$ except for finitely many n, then

$$\begin{aligned} d(\psi_n,\psi_{n+1}) &\leq d(\psi_n,\psi^*) + d(\psi^*,\psi_{n+1}), \text{ and} \\ ||d(\psi_n,\psi_{n+1})|| &\leq K||d(\psi_n,\psi^*)|| + K||d(\psi^*,\psi_{n+1})||. \end{aligned}$$

So $d(\psi_n, \psi_{n+1}) = 0$, for atleast one n. This is contradiction, because $d(\psi_n, \psi_{n+1}) \neq 0$, $\forall n$. Therefore $d(f(\psi^*), \psi^*) = 0$, and ψ^* is a QFP. If f have two QFPs ψ and τ and $d(\psi, \tau) \neq 0$, then

$$\begin{aligned} d(\tau, \psi) &\leq d(\tau, f(\tau)) + d(f(\tau), f(\psi)) + d(f(\psi), \psi) \\ &\leq q[d(f(\tau), \tau) + d(f(\psi), \psi)] = 0 \end{aligned}$$

So $d(\psi, \tau) = 0$ and f has a unique QFP. □

The next theorem consider Chatterjea's contractions [2] for QFPs in SCMSs.

Theorem 3.3. *Let (S, d) be an order complete SCMS, and P be a NCWNC K. Let f be a given function on S to itself such that $d(f(\psi), f(\tau)) \leq q[d(f(\psi), \tau) + d(f(\tau), \psi)]$, $\forall \psi, \tau \in S$, satisfying $d(\psi, \tau) \neq 0$, for some $q \in (0, \frac{1}{2})$. Then f has a unique QFP ψ^* in S.*

Proof. Define $\psi_{n+1} = f(\psi_n)$, where $n = 0, 1, 2, 3, ...$, and ψ_0 is fixed in S. If $d(\psi_{n+1}, \psi_n) = 0$, for some $n = 0, 1, 2, 3, ...$, then ψ_n is a QFP. Suppose $d(\psi_{n+1}, \psi_n) \neq 0$, for every $n = 0, 1, 2, 3, ...$ Then, with $l = (\frac{q}{1-q})$ we have

$$\begin{aligned} d(\psi_{n+1}, \psi_n) = d(f(\psi_n), f(\psi_{n-1})) &\leq q[d(f(\psi_n), \psi_{n-1}) + d(f(\psi_{n-1}), \psi_n)] \\ &= q[d(\psi_{n+1}, \psi_{n-1}) + d(\psi_n, \psi_n)] \\ &\leq q[d(\psi_{n+1}, \psi_n) + d(\psi_n, \psi_{n-1})] \\ d(\psi_{n+1}, \psi_n) &\leq \Big(\frac{q}{1-q}\Big) d(\psi_n, \psi_{n-1}) \\ &= l d(\psi_n, \psi_{n-1}) \\ &\leq l^2 d(\psi_{n-1}, \psi_{n-2}) \leq ... \leq l^n d(\psi_1, \psi_0). \end{aligned}$$

For $m > n \geq 1$, we have

$$\begin{aligned} d(\psi_m, \psi_n) &\leq d(\psi_m, \psi_{m-1}) + d(\psi_{m-1}, \psi_{m-2}) + ... + d(\psi_{n+1}, \psi_n) \\ &\leq (l^{m-1} + l^{m-2} + ... + l^n) d(\psi_1, \psi_0) \\ &\leq \Big(\frac{l^n}{1-l}\Big) d(\psi_1, \psi_0). \end{aligned}$$

We get $\|d(\psi_m, \psi_n)\| \leq (\frac{l^n}{1-l}) K \|d(\psi_1, \psi_0)\|$. This implies that $d(\psi_n, \psi_m) \to 0$ as $n, m \to \infty$. By Remark 2.4, there is a $\psi^* \in S$ such that $d(\psi_n, \psi^*) \to 0$ as $n \to \infty$.

Also, when $d(\psi_n, \psi^*) \neq 0$, we have

$$\begin{aligned} d(f(\psi^*), \psi^*) &\leq d(f(\psi^*), f(\psi_n)) + d(f(\psi_n), \psi^*) \\ &\leq q[d(f(\psi^*), \psi_n) + d(f(\psi_n), \psi^*)] + d(\psi_{n+1}, \psi^*) \\ &\leq q[d(f(\psi^*), \psi^*) + d(\psi_n, \psi^*) + d(\psi_{n+1}, \psi^*)] + d(\psi_{n+1}, \psi^*) \\ d(f(\psi^*), \psi^*) &\leq l[d(\psi_n, \psi^*) + d(\psi_{n+1}, \psi^*)] + \left(\frac{1}{1-q}\right) d(\psi_{n+1}, \psi^*), \text{ and} \\ ||d(f(\psi^*), \psi^*)|| &\leq K||l[d(\psi_n, \psi^*) + d(\psi_{n+1}, \psi^*)] + \left(\frac{1}{1-q}\right) d(\psi_{n+1}, \psi^*)||. \end{aligned}$$

If $d(\psi_n, \psi^*) \neq 0$, for infinitely many n, then $d(\psi^*, f(\psi^*)) = 0$. If $d(\psi_n, \psi^*) = 0$, except for finitely many n, then

$$\begin{aligned} d(\psi_n, \psi_{n+1}) &\leq d(\psi_n, \psi^*) + d(\psi^*, \psi_{n+1}) \text{ and} \\ ||d(\psi_n, \psi_{n+1})|| &\leq K||d(\psi_n, \psi^*)|| + K||d(\psi^*, \psi_{n+1})||. \end{aligned}$$

So $d(\psi_n, \psi_{n+1}) = 0$, for atleast one n. We have contradiction, because $d(\psi_n, \psi_{n+1}) \neq 0$, $\forall n$. Therefore $d(f(\psi^*), \psi^*) = 0$, and ψ^* is a QFP. If f has two QFPs ψ and τ, and $d(\psi, \tau) \neq 0$, then

$$\begin{aligned} d(\tau, \psi) &\leq d(\tau, f(\tau)) + d(f(\tau), f(\psi)) + d(f(\psi), \psi) \\ &\leq q[d(f(\tau), \psi) + d(f(\psi), \tau)] \\ &\leq q[d(f(\tau), \tau) + d(\tau, \psi) + d(f(\psi), \psi) + d(\psi, \tau)]. \\ \text{So, } d(\tau, \psi) &\leq l^m d(\tau, \psi), \text{ for every } m = 1, 2, 3, ... \\ ||d(\tau, \psi)|| &\leq l^m K||d(\tau, \psi)||, \text{ for every } m = 1, 2, 3, ... \end{aligned}$$

Since $l^m K \to 0$ as $m \to \infty$, we get $d(\psi, \tau) = 0$ and f has a unique QFP. □

Theorem 3.4. *Let (S, d) be an order complete SCMS, and P be a NCWNC K. Let f be a given function on S to itself such that $d(f(\psi), f(\tau)) \leq qd(\psi, \tau) + rd(\tau, f(\psi))$, $\forall \psi, \tau \in S$, satisfying $d(\psi, \tau) \neq 0$, for some $q, r \in (0, 1)$ satisfying $\frac{q+r}{1-r} < 1$. Then f has a unique QFP ψ^* in S.*

Proof. Define $\psi_{n+1} = f(\psi_n)$, where $n = 0, 1, 2, 3, ...$, and ψ_0 is fixed in S. If $d(\psi_{n+1}, \psi_n) = 0$, for some $n = 0, 1, 2, 3, ...$, then ψ_n is a QFP. Suppose $d(\psi_{n+1}, \psi_n) \neq 0$, for every $n = 0, 1, 2, 3, ...$ Then with $l = \frac{q+r}{1-r} < 1$ we have

$$\begin{aligned} d(\psi_{n+1}, \psi_n) = d(f(\psi_n), f(\psi_{n-1})) &\leq qd(\psi_n, \psi_{n-1}) + rd(\psi_{n-1}, f(\psi_n)) \\ &\leq qd(\psi_n, \psi_{n-1}) + rd(\psi_{n-1}, \psi_{n+1}) \\ &\leq qd(\psi_n, \psi_{n-1}) + rd(\psi_{n-1}, \psi_n) + rd(\psi_n, \psi_{n+1}), \\ \text{and } d(\psi_{n+1}, \psi_n) &\leq \left(\frac{q+r}{1-r}\right) d(\psi_n, \psi_{n-1}) \\ &= ld(\psi_n, \psi_{n-1}), \\ &\leq l^2 d(\psi_{n-1}, \psi_{n-2}) \leq ... \leq l^n d(\psi_1, \psi_0). \end{aligned}$$

For $m > n \geq 1$, we have

$$\begin{aligned} d(\psi_m, \psi_n) &\leq d(\psi_m, \psi_{m-1}) + d(\psi_{m-1}, \psi_{m-2}) + \ldots + d(\psi_{n+1}, \psi_n) \\ &\leq (l^{m-1} + l^{m-2} + \ldots + l^n) d(\psi_1, \psi_0) \\ &\leq \Big(\frac{l^n}{1-l}\Big) d(\psi_1, \psi_0). \end{aligned}$$

We get $\|d(\psi_m, \psi_n)\| \leq (\frac{l^n}{1-l}) K \|d(\psi_1, \psi_0)\|$. This implies that $d(\psi_n, \psi_m) \to 0$ as $n, m \to \infty$. By Remark 2.4, there is a $\psi^* \in S$ such that $d(\psi_n, \psi^*) \to 0$ as $n \to \infty$. Also, when $d(\psi_n, \psi^*) \neq 0$, we have

$$\begin{aligned} d(f(\psi^*), \psi^*) &\leq d(f(\psi^*), f(\psi_n)) + d(f(\psi_n), \psi^*) \\ &\leq qd(\psi^*, \psi_n) + rd(\psi_n, f(\psi^*)) + d(\psi_{n+1}, \psi^*) \\ &\leq qd(\psi^*, \psi_n) + rd(\psi_n, \psi^*) + rd(\psi^*, f(\psi^*)) + d(\psi_{n+1}, \psi^*), \text{ and} \\ \text{hence } d(f(\psi^*), \psi^*) &\leq ld(\psi_n, \psi^*) + \Big(\frac{1}{1-r}\Big) d(\psi_{n+1}, \psi^*) \text{ so that} \\ ||d(f(\psi^*), \psi^*)|| &\leq lK||d(\psi_n, \psi^*)|| + \Big(\frac{1}{1-r}\Big) K||d(\psi_{n+1}, \psi^*)||. \end{aligned}$$

If $d(\psi_n, x^*) \neq 0$, for infinitely many n, then $d(\psi^*, f(\psi^*)) = 0$. If $d(\psi_n, \psi^*) = 0$, except for finitely many n, then

$$\begin{aligned} d(\psi_n, \psi_{n+1}) &\leq d(\psi_n, \psi^*) + d(\psi^*, \psi_{n+1}) \text{ and} \\ ||d(\psi_n, \psi_{n+1})|| &\leq K||d(\psi_n, \psi^*)|| + K||d(\psi^*, \psi_{n+1})||. \end{aligned}$$

So $d(\psi_n, \psi_{n+1}) = 0$, for atleast one n. This is a contradiction, because $d(\psi_n, \psi_{n+1}) \neq 0$, $\forall n$. Therefore, $d(f(\psi^*), \psi^*) = 0$, and ψ^* is a QFP. If f has two QFPs ψ and τ, and $d(\psi, \tau) \neq 0$ then

$$\begin{aligned} d(\tau, \psi) &\leq d(\tau, f(\tau)) + d(f(\tau), f(\psi)) + d(f(\psi), \psi) \\ &\leq qd(\tau, \psi) + rd(\psi, f(\tau)) \\ &\leq qd(\tau, \psi) + rd(\psi, \tau) + rd(\tau, f(\tau)) \\ &= td(\tau, \psi), \text{ where } t = (q + r), \text{ and hence} \\ d(\tau, \psi) &\leq t^m d(\tau, \psi), \text{ for every } m = 1, 2, 3, ... \\ ||d(\tau, \psi)|| &\leq t^m K||d(\tau, \psi)||, \text{ for every } m = 1, 2, 3, \ldots. \end{aligned}$$

Since $t^m K \to 0$ as $m \to \infty$, we get $d(\psi, \tau) = 0$ and f has a unique QFP. □

Theorem 3.5. *Let (S, d) be an order complete SCMS, and P be a NCWNC K. Let f be a given function on S to itself such that $d(f(\psi), f(\tau)) \leq q[d(\psi, \tau) + d(f(\psi), \psi) + d(f(\tau), \tau) + d(f(\psi), \tau) + d(f(\tau), \psi)]$, $\forall \psi, \tau \in S$, satisfying $d(\psi, \tau) \neq 0$, for some $q \in (0, \frac{1}{5})$. Then f has a unique QFP ψ^* in S.*

PROOF. Define $\psi_{n+1} = f(\psi_n)$, where $n = 0, 1, 2, 3, ...$, and ψ_0 is fixed in S. If $d(\psi_{n+1}, \psi_n) = 0$, for some $n = 0, 1, 2, 3, ...$, then ψ_n is a QFP. Suppose $d(\psi_{n+1}, \psi_n) \neq 0$, for every $n = 0, 1, 2, 3, \ldots$. Then, with $l = (\frac{3q}{1-2q})$ we have

$$\begin{aligned}
d(\psi_{n+1}, \psi_n) &= d(f(\psi_n), f(\psi_{n-1})) \\
&\leq q[d(\psi_n, \psi_{n-1}) + d(f(\psi_n), \psi_n) + d(f(\psi_{n-1}), \psi_{n-1}) + d(f(\psi_n), \psi_{n-1}) \\
&\quad +d(f(\psi_{n-1}), \psi_n)] \\
&= q[d(\psi_n, \psi_{n-1}) + d(\psi_{n+1}, \psi_n) + d(\psi_n, \psi_{n-1}) + d(\psi_{n+1}, \psi_{n-1}) \\
&\quad +d(\psi_n, \psi_n)] \\
d(\psi_{n+1}, \psi_n) &\leq \Big(\frac{3q}{1-2q}\Big) d(\psi_n, \psi_{n-1}) \\
&= l d(\psi_n, \psi_{n-1}) \\
&\leq l^2 d(\psi_{n-1}, \psi_{n-2}) \leq ... \leq l^n d(\psi_1, \psi_0).
\end{aligned}$$

For $m > n \geq 1$, we have

$$\begin{aligned}
d(\psi_m, \psi_n) &\leq d(\psi_m, \psi_{m-1}) + d(\psi_{m-1}, \psi_{m-2}) + ... + d(\psi_{n+1}, \psi_n) \\
&\leq (l^{m-1} + l^{m-2} + ... + l^n) d(\psi_1, \psi_0) \\
&\leq \Big(\frac{l^n}{1-l}\Big) d(\psi_1, \psi_0).
\end{aligned}$$

We get $\|d(\psi_m, \psi_n)\| \leq (\frac{l^n}{1-l}) K \|d(\psi_1, \psi_0)\|$. This implies that $d(\psi_n, \psi_m) \to 0$ as $n, m \to \infty$. By Remark 2.4, there is a $\psi^* \in S$ such that $d(\psi_n, \psi^*) \to 0$ as $n \to \infty$. Also, when $d(\psi_n, \psi^*) \neq 0$, we have

$$\begin{aligned}
d(f(\psi^*), \psi^*) &\leq d(f(\psi^*), f(\psi_n)) + d(f(\psi_n), \psi^*) \\
&\leq q[d(\psi^*, \psi_n) + d(f(\psi^*), \psi^*) + d(f(\psi_n), \psi_n) + d(f(\psi^*), \psi_n) \\
&\quad +d(f(\psi_n), \psi^*)] + d(\psi_{n+1}, \psi^*) \\
d(f(\psi^*), \psi^*) &\leq l[d(\psi^*, \psi_n)] + \Big(\frac{2q}{1-2q}\Big) d(\psi_{n+1}, \psi^*) + \Big(\frac{1}{1-2q}\Big) d(\psi_{n+1}, \psi^*), \text{ and} \\
||d(f(\psi^*), \psi^*)|| &\leq Kl||d(\psi^*, \psi_n)|| + \Big(\frac{2qK}{1-2q}\Big) ||d(\psi_{n+1}, \psi^*)|| \\
&\quad + \Big(\frac{K}{1-2q}\Big) ||d(\psi_{n+1}, \psi^*)||.
\end{aligned}$$

If $d(\psi_n, \psi^*) \neq 0$, for infinitely many n, then $d(\psi^*, f(\psi^*)) = 0$. If $d(\psi_n, \psi^*) = 0$ except for finitely many n, then

$$\begin{aligned}
d(\psi_n, \psi_{n+1}) &\leq d(\psi_n, \psi^*) + d(\psi^*, \psi_{n+1}), \text{ and} \\
||d(\psi_n, \psi_{n+1})|| &\leq K||d(\psi_n, \psi^*)|| + K||d(\psi^*, \psi_{n+1})||.
\end{aligned}$$

So $d(\psi_n, \psi_{n+1}) = 0$, for atleast one n. We have contradiction, because $d(\psi_n, \psi_{n+1}) \neq 0$, $\forall n$. Therefore $d(f(\psi^*), \psi^*) = 0$, and ψ^* is a QFP. If f have two QFPs ψ and τ

and $d(\psi,\tau) \neq 0$, then

$$\begin{aligned} d(\tau,\psi) &\leq d(\tau,f(\tau)) + d(f(\tau),f(\psi)) + d(f(\psi),\psi) \\ &\leq q[d(\tau,\psi) + d(f(\tau),\tau) + d(f(\psi),\psi) + d(f(\tau),\psi) + d(f(\psi),\tau)] \\ &= q[d(\tau,\psi) + d(f(\tau),\tau) + d(\tau,\psi) + d(f(\psi),\psi) + d(\psi,\tau)] \\ \text{So, } d(\tau,\psi) &\leq h^m d(\tau,\psi), \text{ where } h = \frac{q}{1-2q}, \text{ for every } m = 1,2,3,\ldots \\ ||d(\tau,\psi)|| &\leq h^m K ||d(\tau,\psi)||, \text{ for every } m = 1,2,3,\ldots. \end{aligned}$$

Since $h^m K \to 0$ as $m \to \infty$, we get $d(\psi,\tau) = 0$ and f has a unique QFP. □

Remark 3.3. Let (S,d) be an order complete SCMS, and P be a NCWNC K. Let f be a given function on S to itself such that $d(f(\psi),f(\tau)) \leq q[d(\psi,\tau) + d(f(\psi),\psi) + d(f(\tau),\tau)]$, $\forall \psi,\tau \in S$, satisfying $d(\psi,\tau) \neq 0$, for some $q \in (0,\frac{1}{3})$. Then f has a unique QFP ψ^* in S.

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