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Anti complex fuzzy Lie subalgebras under s-norms

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Abstract. The objectives of this article are to present the notions of anti complex fuzzy subalgebras and anti complex fuzzy ideals of Lie algebras under S-norms and we prove that the level subset of them are also subalgebras and ideals of Lie algebras, respectively. Next, we introduce the union and summation of them and we prove that the union and summation of them are also anti complex fuzzy subalgebras and anti complex fuzzy ideals of Lie algebras under S-norms, respectively. Finally, we investigate the homomorphic image (pre-image) of them under Lie homomorphisms.

1. Introduction

Sophus Lie (1842-1899) introduced Lie algebras. The branch of mathematics related to fuzzy set theory is known as fuzzy mathematics. In 1965, It was innovated after the seminal paper of Zadeh [18] on fuzzy sets, Who is the founder of this theory. A new concept of complex fuzzy sets was presented by Ramot et al. [3]. The extension of fuzzy sets to complex fuzzy sets is comparable to the extension of real numbers to complex numbers. The development of complex fuzzy sets can be viewed in [1]. Kim and Lee [2] considered the fuzzy Lie subalgebras and fuzzy Lie ideals and Yehia [16, 17] investigated them. The author by using norms investigated some properties of fuzzy algebraic structures[4]-[15]. In this study, we define anti-complex fuzzy subalgebras of Lie algebra L under S-norms($ACFSS(L)$). We investigate their properties of them and characterize them with subalgebras of Lie algebras. Next using S-norms, we introduce anti-complex fuzzy ideals of Lie algebra

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$L(ACFIS(L))$ and we link them with ideals of Lie algebras. Later, we obtain some results about $ACFIS(L)$ and $ACFSS(L)$ under homomorphisms of Lie algebras. Finally, we define the union and sum of the $ACFIS(L)$ and $ACFSS(L)$ and we prove that the union and summation of them are also $ACFIS(L)$ and $ACFSS(L)$, respectively. Finally, we prove that the homomorphic image (pre-image) of them under Lie homomorphisms will be also $ACFIS(L)$ and $ACFSS(L)$, respectively.

2. Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper. For more details we refer the readers to [**3, 4, 5**].

Definition 2.1. A Lie algebra is a vector space L over a field F (equal to R or C) on which $L \times L \to L$ denoted by $(x, y) \to [x, y]$ is defined satisfying the following axioms:
(1) $[x, y]$ is bilinear,
(2) $[x, x] = 0$ for all $x \in L$,
(3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity),
for all $x, y, z \in L$.

In this paper, L will be denoted as a Lie algebra. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x, y], z] = [x, [y, z]]$. But it is anti commutative, i.e., $[x, y] = -[y, x]$. A subspace H of L closed under $[,]$ will be called a Lie subalgebra. A subspace I of L with the property $[I, L] \subseteq I$ will be called a Lie ideal of L. Obviously, any Lie ideal is a subalgebra.

Definition 2.2. Let L_1 and L_2 be Lie algebras over a field F. A linear transformation $f : L_1 \to L_2$ is called a Lie homomorphism if $f([x, y]) = [f(x), f(y)]$ for all $x, y \in L_1$.

Definition 2.3. Let X be a nonempty set. A complex fuzzy set A on X is an object having the form $A = \{(x, \mu_A(x))|x \in X\}$, where μ_A denotes the degree of membership function that assigns each element $x \in X$ a complex number $\mu_A(x))$ lies within the unit circle in the complex plane. We shall assume that is $\mu_A(x))$ will be represented by $r_A(x)e^{iw_A(x)}$ where $i = \sqrt{-1}$, and $r : X \to [0, 1]$ and $w : X \to [0, 2\pi]$. Note that by setting $w(x) = 0$ in the definition above, we return back to the traditional fuzzy subset. Let $\mu_1 = r_1e^{w_1}$, and $\mu_2 = r_2e^{w_2}$ be two complex numbers lie within the unit circle in the complex plane. By $\mu_1 \leq \mu_2$, we mean $r_1 \leq r_2$ and $w_1 \leq w_2$.

Definition 2.4. An s-norm S is a function $S : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(1) $S(x, 0) = x$,
(2) $S(x, y) \leq S(x, z)$ if $y \leq z$,

(3) $S(x, y) = S(y, x)$,
(4) $S(x, S(y, z)) = S(S(x, y), z)$,
for all $x, y, z \in [0, 1]$.

We say that S is idempotent if for all $x \in [0, 1]$, $S(x, x) = x$.

Example 2.5. The basic S-norms are

$$S_m(x, y) = \max\{x, y\},$$

$$S_b(x, y) = \min\{1, x + y\}$$

and

$$S_p(x, y) = x + y - xy$$

for all $x, y \in [0, 1]$, here, S_m is the standard union, S_b is the bounded sum, S_p is the algebraic sum.

Lemma 2.1. *Let S be a s-norm. Then*

$$S(S(x, y), S(w, z)) = S(S(x, w), S(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

3. Anti complex fuzzy Lie subalgebras under S-norms

Definition 3.1. Let L be a Lie subalgebra and $\mu : L \to [0, 1]$ be a complex fuzzy set on L. Then $\mu = re^{iw}$ is said to be an anti complex fuzzy subalgebra of L under s-norm S if the following conditions hold:
(1) $r(x + y) \leq S(r(x), r(y))$,
(2) $r(\alpha x) \leq r(x)$,
(3) $r([x, y]) \leq S(r(x), r(y))$,
(4) $w(x + y) \leq \max\{w(x), w(y)\}$,
(5) $w(\alpha x) \leq w(x)$,
(6) $w([x, y]) \leq \max\{w(x), w(y)$,
for all $x, y \in L$ and $\alpha \in F$.
Denote by $ACFSS(L)$, the set of all anti complex fuzzy subgroups of L under s-norm S.

Definition 3.2. Let L be a Lie subalgebra and $\mu : L \to [0, 1]$ be a complex fuzzy set on L. Then $\mu = re^{iw}$ is said to be an anti complex fuzzy Lie ideal of L under s-norm S if the following conditions hold:
(1) $r(x + y) \leq S(r(x), r(y))$,
(2) $r(\alpha x) \leq r(x)$,
(3) $r([x, y]) \leq r(x)$,
(4) $w(x + y) \leq \max\{w(x), w(y)\}$,
(5) $w(\alpha x) \leq w(x)$,

(6) $w([x,y]) \leq w(x)$,
for all $x, y \in L$ and $\alpha \in F$.
Denote by $ACFIS(L)$, the set of all anti complex fuzzy Lie ideals of L under s-norm S.

Example 3.3. It is well known that the set $\mathbf{R}^3 = \{(x,y,z) : x,y,z \in \mathbf{R}\}$ of all 3-dimensional real vectors forms a Lie algebra over $F = \mathbf{R}$ with the usual cross product $\times$. Define $\mu : \mathbf{R}^3 \to \mathbf{E}^2$, where ($\mathbf{E}^2$ is the unit disc), by

$$\mu(x,y,z) = \begin{cases} 0 & \text{if } x = y = z = 0 \\ 0.45e^{i\frac{\pi}{2}} & \text{if } x = y = 0 \textit{ and } z \neq 0 \\ 0.75e^{i\frac{3\pi}{4}} & \text{otherwise} \end{cases}$$

if $S(a,b) = a + b - ab$ for all $a, b \in [0,1]$, then $\mu \in ACFSS(L)$. Let $x = (0,0,2) \in L = \mathbf{R}^3$ and $y = (1,0,0) \in L = \mathbf{R}^3$ then $[x,y] = (0,1,0)$ and so $r([x,y]) = r(0,-2,0) = 0.75 \nleq 0.45 = r(x)$ thus $\mu \notin ACFIS(L)$.

Lemma 3.1. *Let $\mu \in ACFSS(L)$ and S be an idempotent s-norm.*
(1) $\mu(0) \leq \mu(x)$ for all $x \in L$.
(2) $\mu(x) = \mu(-x)$ for all $x \in L$.
(3) $\mu([x,y]) = \mu([y,x])$ for all $x, y \in L$.
(4) $\mu(x-y) = \mu(0)$ implies that $\mu(x) = \mu(y)$ for all $x, y \in L$.

Proof. (1) Let $x \in L$. As

$$r(0) = r(x + (-x)) \leq S(r(x), r(-x)) \leq S(r(x), r(x)) = r(x)$$

and

$$w(0) = w(x + (-x)) \leq \max\{w(x), w(-x)\} \leq \max\{w(x), w(x)\} = w(x)$$

thus $r(0) \leq r(x)$ and $w(0) \leq w(x)$ thus $\mu(0) = r(0)e^{iw(0)} \leq r(x)e^{iw(x)} = \mu(x)$.
(2) Let $x \in L$. Thus

$$r(-x) = r((-1)x) \leq r(x) = r(-(-x)) \leq r(-x)$$

and

$$w(-x) = w((-1)x) \leq w(x) = w(-(-x)) \leq w(-x)$$

so $r(x) = r(-x)$ and $w(x) = w(-x)$ then $\mu(x) = r(x)e^{iw(x)} = r(-x)e^{iw(-x)} = \mu(-x)$.
(3) Let $x, y \in L$. Then $r([x,y]) = r(-[y,x]) = r([y,x])$ and $w([x,y]) = w(-[y,x]) = w([y,x])$ so

$$\mu([x,y]) = r([x,y])e^{iw([x,y])} = r([y,x])e^{iw([y,x])} = \mu([y,x]).$$

(4) Let $x, y \in L$. Now

$$
\begin{aligned}
r(y) &= r(x-(x-y)) \\
&= r(x+(-(x-y))) \\
&\leq S(r(x), r(-(x-y))) \\
&= S(r(x), r(x-y)) \\
&= S(r(x), r(0)) \\
&\leq S(r(x), r(x)) \\
&= r(x) \\
&= r(x-y+y) \\
&\leq S(r(x-y), r(y)) \\
&= S(r(0), r(y)) \\
&\leq S(r(y), r(y)) \\
&= r(y)
\end{aligned}
$$

thus $r(x) = r(y)$. Also

$$
\begin{aligned}
w(y) &= w(x-(x-y)) \\
&= w(x+(-(x-y))) \\
&\leq \max\{w(x), w(-(x-y))\} \\
&= \max\{w(x), w(x-y)\} \\
&= \max\{w(x), w(0)\} \\
&\leq \max\{w(x), w(x)\} \\
&= w(x) \\
&= w(x-y+y) \\
&\leq \max\{w(x-y), w(y)\} \\
&= \max\{w(0), w(y) \\
&\leq \max\{w(y), w(y)\} \\
&= w(y)
\end{aligned}
$$

thus $w(x) = w(y)$. Therefore $\mu(x) = r(x)e^{iw(x)} = r(y)e^{iw(y)} = \mu(y)$. □

Proposition 3.2. *Let L be a Lie subalgebra and $\mu : L \to [0,1]$ be a complex fuzzy set on L. If S be idempotent s-norm, then the following statements are equivalent:*
(1) $\mu = re^{iw} \in ACFSS(L)$.
(2) The set

$$L^t_\mu = \{x \in L \mid \mu(x) = r(x)e^{iw(x)} \leq t\} = \{x \in L \mid r(x) \leq t, w(x) \leq t\}$$

is a subalgebra of L for every $t \in Im(\mu)$.

Proof. Let $\mu = re^{iw} \in ACFSS(L)$ and $x, y \in L^t_\mu$ and $\alpha \in F$. We must prove that $x+y, \alpha x, [x,y] \in L^t_\mu$. As

$$r(x+y) \leq S(r(x), r(y)) \leq S(t,t) = t,$$

$$w(x+y) \leq \max\{w(x), w(y)\} \leq w\{t,t\} = t,$$

$$r(\alpha x) \leq r(x) \leq t,$$

$$w(\alpha x) \leq w(x) \leq t,$$

$$r([x,y]) \leq S(r(x), r(y)) \leq S(t,t) = t$$

and

$$w([x,y]) \leq \max\{w(x), w(y)\} \leq w\{t,t\} = t,$$

we get that $x+y, \alpha x, [x,y] \in L^t_\mu$. Then L^t_μ will be subalgebra of L for every $t \in Im(\mu)$. Conversely, let L^t_μ be a Lie subalgebra of L for every $t \in Im(\mu)$. Let $x, y \in L$ and $\alpha \in F$. We can say that $\mu(y) \leq \mu(x) = t$ so $r(y) \leq r(x) = t$ and $w(y) \leq w(x) = t$ thus $x, y \in L^t_\mu$ and then $x+y, \alpha x, [x,y] \in L^t_\mu$. Since

$$r(x+y) \leq t = S(r(x), r(y)),$$

$$w(x+y) \leq t = \max\{w(x), w(y)\},$$

$$r(\alpha x) \leq t = r(x),$$

$$w(\alpha x) \leq t = w(x),$$

$$r([x,y]) \leq t = S(r(x), r(y)),$$

and

$$w([x,y]) \leq t = \max\{w(x), w(y)\}.$$

So $\mu = re^{iw} \in ACFSS(L)$. □

Corollary 3.3. *Let L be a Lie subalgebra and $\mu : L \to [0,1]$ be a complex fuzzy set on L. If S be idempotent s-norm, then the following statements are equivalent:*
(1) $\mu = re^{iw} \in ACFSS(L)$.
(2) The set

$$L^t_\mu = \{x \in L \mid \mu(x) = r(x)e^{iw(x)} < t\} = \{x \in L \mid r(x) < t, w(x) < t\}$$

is a subalgebra of L for every $t \in Im(\mu)$.

Proposition 3.4. *Let L be a Lie subalgebra and $\mu : L \to [0,1]$ be a complex fuzzy set on L. If S be idempotent s-norm, then the following statements are equivalent:*
(1) $\mu = re^{iw} \in ACFIS(L)$.
(2) The set

$$L^t_\mu = \{x \in L \mid \mu(x) = r(x)e^{iw(x)} \leq t\} = \{x \in L \mid r(x) \leq t, w(x) \leq t\}$$

is an ideal of L for every $t \in Im(\mu)$.

PROOF. See proof of Proposition 3.2. $\square$

Corollary 3.5. *Let L be a Lie subalgebra and $\mu : L \to [0,1]$ be a complex fuzzy set on L. If S be idempotent s-norm, then the following statements are equivalent:*
(1) $\mu = re^{iw} \in ACFSS(L)$.
(2) The set

$$L^t_\mu = \{x \in L \mid \mu(x) = r(x)e^{iw(x)} < t\} = \{x \in L \mid r(x) < t, w(x) < t\}$$

is an ideal of L for every $t \in Im(\mu)$.

Definition 3.4. Let $f : G \to H$ be a mapping and $\mu_G = r_G e^{iw_G}$ and $\mu_H = r_H e^{iw_H}$ be two complex fuzzy sets on G and H, respectively. Define $f(\mu_G) : H \to [0,1]$ as

$$f(\mu_G) = f(r_G e^{iw_G}) = f(r_G)e^{if(w_G)}$$

such that for all $h \in H$ we define

$$f(r_G)(h) = \inf\{r_G(g) \mid g \in G, f(g) = h\}$$

and

$$f(w_G)(h) = \inf\{w_G(g) \mid g \in G, f(g) = h\}.$$

Also define $f^{-1}(\mu_H) : G \to [0,1]$ as

$$f^{-1}(r_H e^{iw_H}) = f^{-1}(r_H)e^{if^{-1}(w_H)}$$

such that for all $g \in G$ we define

$$f^{-1}(r_H e^{iw_H})(g) = r_H(f(g))e^{iw_H(f(g))}.$$

Proposition 3.6. *Let $\mu_L = r_L e^{iw_L} \in ACFSS(L)$ and $f : L \to M$ be an epimorphism of Lie algebras. Then $f(\mu_L) \in ACFSS(M)$.*

PROOF. (1) Let $m_1, m_2 \in M$ and $l_1, l_2 \in L$ such that $m_1 = f(l_1)$ and $m_2 = f(l_2)$. Then

$$\begin{aligned} f(r_L)(m_1 + m_2) &= \inf\{r_L(l_1 + l_2) \mid l_1, l_2 \in L, f(l_1 + l_2) = m_1 + m_2\} \\ &\leq \inf\{S(r_L(l_1), r_L(l_2)) \mid l_1, l_2 \in L, f(l_1) = m_1, f(l_2) = m_2\} \\ &= S(\inf\{r_L(l_1) \mid l_1 \in L, f(l_1) = m_1\}, \inf\{r_L(l_2) \mid l_2 \in L, f(l_2) = m_2\}) \\ &= S(f(r_L)(m_1), f(r_L)(m_2)) \end{aligned}$$

thus

$$f(r_L)(m_1 + m_2) \leq S(f(r_L)(m_1), f(r_L)(m_2)).$$

Also

$$\begin{aligned} f(w_L)(m_1+m_2) &= \inf\{w_L(l_1+l_2) \mid l_1,l_2 \in L, f(l_1+l_2)=m_1+m_2\} \\ &\le \inf\{\max\{w_L(l_1),w_L(l_2)\} \mid l_1,l_2 \in L, f(l_1)=m_1, f(l_2)=m_2\} \\ &= \max\{\inf\{w_L(l_1) \mid l_1 \in L, f(l_1)=m_1\}, \inf\{w_L(l_2)\} \mid l_2 \in L, f(l_2)=m_2\}) \\ &= \max\{f(r_L)(m_1), f(r_L)(m_2)\} \end{aligned}$$

then

$$f(w_L)(m_1+m_2) \le \max\{f(w_L)(m_1), f(w_L)(m_2)\}.$$

(2) Let $m \in M, \alpha \in F$ and $l \in L$ such that $m = f(l)$. Now

$$f(r_L)(\alpha m) = \inf\{r_L(\alpha l) \mid l \in L, f(\alpha l)=\alpha m\} \le \inf\{r_L(l) \mid l \in L, f(l)=m\} = f(r_L)(m)$$

and

$$f(w_L)(\alpha m) = \inf\{w_L(\alpha l) \mid l \in L, f(\alpha l)=\alpha m\} \le \inf\{w_L(l) \mid l \in L, f(l)=m\} = f(w_L)(m).$$

(3) Let $m_1, m_2 \in M$ and $l_1, l_2 \in L$ such that $m_1 = f(l_1)$ and $m_2 = f(l_2)$. As

$$\begin{aligned} f(r_L)([m_1,m_2]) &= \inf\{r_L([l_1,l_2]) \mid l_1,l_2 \in L, f([l_1,l_2])=[m_1,m_2]\} \\ &\le \inf\{S(r_L(l_1),r_L(l_2)) \mid l_1,l_2 \in L, ([f(l_1),f(l_2)])=[m_1,m_2]\} \\ &= S(\inf\{r_L(l_1) \mid l_1 \in L, f(l_1)=m_1\}, \inf\{r_L(l_2) \mid l_2 \in L, f(l_2)=m_2\}) \\ &= S(f(r_L)(m_1), f(r_L)(m_2)) \end{aligned}$$

then

$$f(r_L)([m_1,m_2]) \le S(f(r_L)(m_1), f(r_L)(m_2)).$$

Moreover

$$\begin{aligned} f(w_L)([m_1,m_2]) &= \inf\{w_L([l_1,l_2]) \mid l_1,l_2 \in L, f([l_1,l_2])=[m_1,m_2]\} \\ &\le \inf\{\max\{w_L(l_1),w_L(l_2)\} \mid l_1,l_2 \in L, ([f(l_1),f(l_2)])=[m_1,m_2]\} \\ &= \max\{\inf\{w_L(l_1) \mid l_1 \in L, f(l_1)=m_1\}, \inf\{w_L(l_2) \mid l_2 \in L, f(l_2)=m_2\}\} \\ &= \max\{f(w_L)(m_1), f(w_L)(m_2)\} \end{aligned}$$

then

$$f(w_L)([m_1,m_2]) \le \max\{f(w_L)(m_1), f(w_L)(m_2)\}.$$

Therefore (1) - (3) mean that $f(\mu_L) = f(r_L e^{iw_L}) = f(r_L)e^{if(w_L)} \in ACFSS(M)$. □

Proposition 3.7. *Let $\mu_L = r_L e^{iw_L} \in ACFIS(L)$ and $f : L \to M$ be an epimorphism of Lie algebras. Then $f(\mu_L) \in ACFIS(M)$.*

PROOF. The proof is similart to the proof of Proposition 3.6. □

Proposition 3.8. *Let $\mu_M = r_M e^{iw_M} \in ACFSS(M)$ and $f : L \to M$ be an epimorphism of Lie algebras. Then $f^{-1}(\mu_M) \in ACFSS(L)$.*

PROOF. (1) Let $l_1, l_2 \in L$. Thus

$$\begin{aligned} f^{-1}(r_M)(l_1+l_2) &= r_M(f(l_1+l_2)) \\ &= r_M(f(l_1)+f(l_2)) \\ &= r_M(f(l_1)+f(l_2)) \\ &\leq S(r_M(f(l_1), r_M(f(l_2)) \\ &= S(f^{-1}(r_M)(l_1), f^{-1}(r_M)(l_2)) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(w_M)(l_1+l_2) &= w_M(f(l_1+l_2)) \\ &= w_M(f(l_1)+f(l_2)) \\ &= w_M(f(l_1)+f(l_2)) \\ &\leq \max\{w_M(f(l_1), w_M(f(l_2)\} \\ &= \max\{f^{-1}(w_M)(l_1), f^{-1}(w_M)(l_2)\} \end{aligned}$$

and so

$$f^{-1}(r_M)(l_1+l_2) \leq S(f^{-1}(r_M)(l_1), f^{-1}(r_M)(l_2))$$

and

$$f^{-1}(w_M)(l_1+l_2) \leq \max\{f^{-1}(w_M)(l_1), f^{-1}(w_M)(l_2)\}.$$

(2) Let $l_1, l_2 \in L$. Now

$$\begin{aligned} f^{-1}(r_M)([l_1, l_2]) &= r_M(f([l_1, l_2])) \\ &= r_M([f(l_1), f(l_2)]) \\ &\leq S(r_M(f(l_1), r_M(f(l_2)) \\ &= S(f^{-1}(r_M)(l_1), f^{-1}(r_M)(l_2)) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(w_M)([l_1, l_2]) &= w_M(f([l_1, l_2])) \\ &= w_M([f(l_1), f(l_2)]) \\ &\leq \max\{w_M(f(l_1), w_M(f(l_2)\} \\ &= \max\{f^{-1}(w_M)(l_1), f^{-1}(w_M)(l_2)\} \end{aligned}$$

then

$$f^{-1}(r_M)([l_1, l_2]) \leq S(f^{-1}(r_M)(l_1), f^{-1}(r_M)(l_2))$$

and

$$f^{-1}(w_M)([l_1, l_2]) \leq \max\{f^{-1}(w_M)(l_1), f^{-1}(w_M)(l_2)\}.$$

(3) Let $l \in L$ and $\alpha \in F$. Then

$$f^{-1}(r_M)(\alpha l) = r_M(f(\alpha l)) = r_M(\alpha f(l)) \leq r_M(f(l)) = f^{-1}(r_M)(l)$$

and

$$f^{-1}(w_M)(\alpha l) = w_M(f(\alpha l)) = w_M(\alpha f(l)) \leq w_M(f(l)) = f^{-1}(w_M)(l).$$

Then from (1)-(3) we get that $f^{-1}(\mu_M) \in ACFSS(L)$. □

Proposition 3.9. *Let $\mu_M = r_M e^{iw_M} \in ACFIS(M)$ and $f : L \to M$ be an epimorphism of Lie algebras. Then $f^{-1}(\mu_M) \in ACFIS(L)$.*

PROOF. It is similar to the proof of Proposition 3.8. □

Definition 3.5. Let $\mu_1 = r_1 e^{iw_1}$ and $\mu_2 = r_2 e^{iw_2}$ be two complex fuzzy sets on L. Define the union $\mu_1 \cup \mu_2$ as

$$\mu_1 \cup \mu_2 = r_1 e^{iw_1} \cup r_2 e^{iw_2} = (r_1 \cup r_2) e^{i(w_1 \cup w_2)}$$

such that $r_1 \cup r_2 : L \to [0,1]$ and $w_1 \cup w_2 : L \to [0, 2\pi]$ and for all $x \in L$ define

$$(r_1 \cup r_2)(x) = S(r_1(x), r_2(x))$$

and

$$(w_1 \cup w_2)(x) = \max\{w_1(x), w_2(x)\}.$$

Proposition 3.10. *Let $\mu_1 = r_1 e^{iw_1} \in ACFSS(L)$ and $\mu_2 = r_2 e^{iw_2} \in ACFSS(L)$. Then $\mu_1 \cup \mu_2 \in ACFSS(L)$.*

PROOF. Let $x, y \in L$ and $\alpha \in F$. Then
(1)

$$\begin{aligned} (r_1 \cup r_2)(x+y) &= S(r_1(x+y), r_2(x+y)) \\ &\leq S(S(r_1(x), r_1(y)), S(r_2(x), r_2(y))) \\ &= S(S(r_1(x), r_2(x)), S(r_1(y), r_2(y))) \\ &= S((r_1 \cup r_2)(x), (r_1 \cup r_2)(y)) \end{aligned}$$

and

$$\begin{aligned} (w_1 \cup w_2)(x+y) &= \max\{w_1(x+y), w_2(x+y)\} \\ &\leq \max\{\max\{w_1(x), w_1(y)\}, \max\{w_2(x), w_2(y)\}\} \\ &= \max\{\max\{w_1(x), w_2(x)\}, \max\{w_1(y), w_2(y)\}\} \\ &= \max\{(w_1 \cup w_2)(x), (w_1 \cup w_2)(y)\} \end{aligned}$$

thus

$$(r_1 \cup r_2)(x+y) \leq S((r_1 \cup r_2)(x), (r_1 \cup r_2)(y))$$

and

$$(w_1 \cup w_2)(x+y) \leq \max\{(w_1 \cup w_2)(x), (w_1 \cup w_2)(y)\}.$$

(2)

$$\begin{aligned}(r_1 \cup r_2)([x,y]) &= S(r_1([x,y]), r_2([x,y]))\\ &\leq S(S(r_1(x), r_1(y)), S(r_2(x), r_2(y)))\\ &= S(S(r_1(x), r_2(x)), S(r_1(y), r_2(y)))\\ &= S((r_1 \cup r_2)(x), (r_1 \cup r_2)(y))\end{aligned}$$

so

$$(r_1 \cup r_2)([x,y] \leq S((r_1 \cup r_2)(x), (r_1 \cup r_2)(y)).$$

Also

$$\begin{aligned}(w_1 \cup w_2)([x,y]) &= \max\{w_1([x,y]), w_2([x,y])\}\\ &\leq \max\{\max\{w_1(x), w_1(y)\}, \max\{w_2(x), w_2(y)\}\}\\ &= \max\{\max\{w_1(x), w_2(x)\}, \max\{w_1(y), w_2(y)\}\}\\ &= \max\{(w_1 \cup w_2)(x), (w_1 \cup w_2)(y)\}\end{aligned}$$

thus

$$(w_1 \cup w_2)([x,y]) \leq \max\{(w_1 \cup w_2)(x), (w_1 \cup w_2)(y)\}.$$

(3)

$$(r_1 \cup r_2)(\alpha x) = S(r_1(\alpha x), r_2(\alpha x)) \leq S(r_1(x), r_2(x)) = (r_1 \cup r_2)(x)$$

and

$$(w_1 \cup w_2)(\alpha x) = \max\{w_1(\alpha x), w_2(\alpha x)\} \leq \max\{w_1(x), w_2(x)\} = (w_1 \cup w_2)(x).$$

Thus (1)-(3) give us that $\mu_1 \cup \mu_2 \in ACFSS(L)$. □

Proposition 3.11. *Let* $\mu_1 = r_1 e^{iw_1} \in ACFIS(L)$ *and* $\mu_2 = r_2 e^{iw_2} \in ACFIS(L)$. *Then* $\mu_1 \cup \mu_2 \in ACFIS(L)$.

Proof. It is similar to the proof of Proposition 3.10. □

Definition 3.6. Let $\mu_1 = r_1 e^{iw_1}$ and $\mu_2 = r_2 e^{iw_2}$ be two complex fuzzy sets on L. Define the sum $\mu_1 + \mu_2$ as

$$\mu_1 + \mu_2 = r_1 e^{iw_1} + r_2 e^{iw_2} = (r_1 + r_2) e^{i(w_1 + w_2)}$$

such that $r_1 + r_2 : L \to [0,1]$ and $w_1 + w_2 : L \to [0, 2\pi]$ and for all $x \in L$ define

$$(r_1 + r_2)(x) = \inf_{x=a+b} S(r_1(a), r_2(b))$$

and

$$(w_1 + w_2)(x) = \max_{x=a+b}\{w_1(a), w_2(b)\}.$$

Proposition 3.12. *Let* $\mu_1 = r_1 e^{iw_1} \in ACFSS(L)$ *and* $\mu_2 = r_2 e^{iw_2} \in ACFSS(L)$. *Then* $\mu_1 + \mu_2 \in ACFSS(L)$.

PROOF. Let $x, y, a, b, c, d \in L$ and $\alpha \in F$.
(1)

$$\begin{aligned}(r_1+r_2)(x+y) &= \inf_{x+y=a+b+c+d} S(r_1(a+b), r_2(c+d)) \\ &\leq \inf_{x+y=a+b+c+d} S(S(r_1(a), r_1(b)), S(r_2(c), r_2(d))) \\ &= \inf_{x+y=a+c+b+d} S(S(r_1(a), r_2(c)), S(r_1(b), r_2(d))) \\ &= S(\inf_{x=a+c} S(r_1(a), r_2(c)), \inf_{y=b+d} S(r_1(b), r_2(d))) \\ &= S((r_1+r_2)(x), (r_1+r_2)(y))\end{aligned}$$

and then

$$(r_1+r_2)(x+y) \leq S((r_1+r_2)(x), (r_1+r_2)(y)).$$

Also

$$\begin{aligned}(w_1+w_2)(x+y) &= \max_{x+y=a+b+c+d} \{w_1(a+b), w_2(c+d)\} \\ &\leq \max_{x+y=a+b+c+d} \{\max\{w_1(a), w_1(b)\}, \max\{w_2(c), w_2(d)\}\} \\ &= \max_{x+y=a+c+b+d} \{\max\{w_1(a), w_2(c)\}, \max\{w_1(b), w_2(d)\}\} \\ &= \max\{\min_{x=a+c} \{w_1(a), w_2(c)\}, \max_{y=b+d} \{w_1(b), w_2(d)\}\} \\ &= \max\{(w_1+w_2)(x), (w_1+w_2)(y)\}\end{aligned}$$

and then

$$(w_1+w_2)(x+y) \leq \max\{(w_1+w_2)(x), (w_1+w_2)(y)\}.$$

(2)

$$\begin{aligned}(r_1+r_2)([x,y]) &= \inf_{[x,y]=[a,b]+[c,d]} S(r_1([a,b]), r_2([c,d])) \\ &\leq \inf_{[x,y]=[a,b]+[c,d]} S(S(r_1(a), r_1(b)), S(r_2(c), r_2(d))) \\ &= \inf_{[x,y]=[a+c,b+d]} S(S(r_1(a), r_2(c)), S(r_1(b), r_2(d))) \\ &= S(\inf_{x=a+c} S(r_1(a), r_2(c)), \inf_{y=b+d} S(r_1(b), r_2(d))) \\ &= S((r_1+r_2)(x), (r_1+r_2)(y))\end{aligned}$$

so

$$(r_1+r_2)([x,y]) \leq S((r_1+r_2)(x), (r_1+r_2)(y)).$$

Also

$$\begin{aligned}(w_1+w_2)([x,y]) &= \max_{[x,y]=[a,b]+[c,d]}\{w_1([a,b]),w_2([c,d])\}\\ &\le \max_{[x,y]=[a,b]+[c,d]}\{\max\{w_1(a),w_1(b)\},\max\{w_2(c),w_2(d)\}\}\\ &= \max_{[x,y]=[a+c,b+d]}\{\max\{w_1(a),w_2(c)\},\max\{w_1(b),w_2(d)\}\}\\ &= \max\{\max_{x=a+c}\{w_1(a),w_2(c)\},\max_{y=b+d}\{w_1(b),w_2(d)\}\}\\ &= \max\{(w_1+w_2)(x),(w_1+w_2)(y)\}.\end{aligned}$$

Thus

$$(w_1+w_2)(x+y)\le \max\{(w_1+w_2)(x),(w_1+w_2)(y)\}.$$

(3)

$$(r_1+r_2)(\alpha x)=\inf_{\alpha x=\alpha a+\alpha b} S(r_1(\alpha a),r_2(\alpha b))\le \inf_{x=a+b} S(r_1(a),r_2(b))=(r_1+r_2)(x)$$

and

$$(w_1+w_2)(\alpha x)=\max_{\alpha x=\alpha a+\alpha b}\{w_1(\alpha a),w_2(\alpha b)\}\le \max_{x=a+b}\{w_1(a),w_2(b)\}=(w_1+w_2)(x).$$

Then from (1)-(3) we get that $\mu_1+\mu_2\in ACFSS(L)$. □

Proposition 3.13. *Let* $\mu_1=r_1e^{iw_1}\in ACFIS(L)$ *and* $\mu_2=r_2e^{iw_2}\in ACFIS(L)$. *Then* $\mu_1+\mu_2\in ACFIS(L)$.

Proof. It is similar to the proof of Proposition 3.12. □

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