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On various properties of module Lau product of algebras

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Abstract. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$ be complex algebras, $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ be an algebra homomorphism, and let $\mathcal{A}$ be an $\mathcal{X}$-bimodule. We define a product on $\mathcal{A} \times \mathcal{B}$ as $(a_1, b_1)(a_2, b_2) = (a_1a_2 + a_1 \cdot \theta(b_2) + \theta(b_1) \circ a_2, b_1b_2)$ for all $(a_1, b_1), (a_2, b_2) \in \mathcal{A} \times \mathcal{B}$ and write $\mathcal{A} \times \mathcal{B}$ with this product by $\mathcal{A} \times_\theta \mathcal{B}$. We shall study some basic properties of $\mathcal{A} \times_\theta \mathcal{B}$. When $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{X}$ are Banach algebras, $\mathcal{A}$ is a Banach $\mathcal{X}$-bimodule, and θ is a continuous homomorphism with the norm at most 1, we determine the ideals of $\mathcal{A} \times_\theta \mathcal{B}$ of a certain type, the Gelfand space of this Banach algebra, and the module multipliers of this Banach algebra.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{X}$ be complex algebras. An algebra $\mathcal{A}$ is a *left* $\mathcal{X}-$*module* if there exists a bilinear map $(\alpha, a) \in \mathcal{X} \times \mathcal{A} \mapsto \alpha \circ a \in \mathcal{A}$ satisfying $(\alpha\beta) \circ a = \alpha \circ (\beta \circ a)$ and $\alpha \circ (ab) = (\alpha \circ a)b$ for all $\alpha, \beta \in \mathcal{X}$ and $a, b \in \mathcal{A}$. It is a *right* $\mathcal{X}-$*module* if there exists a bilinear map $(a, \alpha) \in \mathcal{A} \times \mathcal{X} \mapsto a \cdot \alpha \in \mathcal{A}$ satisfying $(ab) \cdot \alpha = a(b \cdot \alpha)$ and $a \cdot (\alpha\beta) = (a \cdot \alpha) \cdot \beta$ for all $\alpha, \beta \in \mathcal{X}$ and $a, b \in \mathcal{A}$. It is a $\mathcal{X}-$*bimodule* if it is both left $\mathcal{X}-$module, right $\mathcal{X}-$module, $\alpha \circ (a \cdot \beta) = (\alpha \circ a) \cdot \beta$ and $(a \cdot \alpha)b = a(\alpha \circ b)$ for all $\alpha, \beta \in \mathcal{X}$ and $a, b \in \mathcal{A}$. It is a *symmetric* $\mathcal{X}-$*bimodule* if it is $\mathcal{X}-$bimodule, $\alpha \circ a = a \cdot \alpha$ for all $a \in \mathcal{A}$ and $\alpha \in \mathcal{X}$.

Let $(\mathcal{A}, \|\cdot\|_\mathcal{A})$ and $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ be normed algebras. Then $\mathcal{A}$ is a *normed left* $\mathcal{X}-$*module* if it is a left $\mathcal{X}-$module and there exists a constant $P > 0$ such that $\|\alpha \circ a\|_\mathcal{A} \leq P\|\alpha\|_\mathcal{X}\|a\|_\mathcal{A}$ for all $\alpha \in \mathcal{X}$ and $a \in \mathcal{A}$. It is a *normed right* $\mathcal{X}-$*module* if it is a

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right $\mathcal{X}-$module and there exists a constant $Q > 0$ such that $\|a\cdot\alpha\|_{\mathcal{A}} \leq Q\|a\|_{\mathcal{A}}\|\alpha\|_{\mathcal{X}}$ for all $a \in \mathcal{A}$ and $\alpha \in \mathcal{X}$. It is a *normed* $\mathcal{X}-$*bimodule* if it is an $\mathcal{X}-$bimodule and there is $R > 0$ such that $\|\alpha \circ a\|_{\mathcal{A}} \leq R\|\alpha\|_{\mathcal{X}}\|a\|_{\mathcal{A}}$ and $\|a \cdot \alpha\|_{\mathcal{A}} \leq R\|a\|_{\mathcal{A}}\|\alpha\|_{\mathcal{X}}$ for all $a \in \mathcal{A}$ and $\alpha \in \mathcal{X}$. It is a *Banach* $\mathcal{X}-$*bimodule* if both $\mathcal{A}$ and $\mathcal{X}$ are complete as a normed linear space.

Definition 1.1. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$ be complex algebras, $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ be an algebra homomorphism, and let $\mathcal{A}$ be an $\mathcal{X}-$bimodule. We define a product on $\mathcal{A} \times \mathcal{B}$ as

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + a_1 \cdot \theta(b_2) + \theta(b_1) \circ a_2, b_1b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in \mathcal{A} \times \mathcal{B}$. Then $\mathcal{A} \times \mathcal{B}$ together with co-ordinatewise linear operations and the above product is an associative algebra. We denote this algebra by $\mathcal{A} \times_\theta \mathcal{B}$.

If $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, and $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ are Banach algebras, $\mathcal{A}$ is a Banach $\mathcal{X}-$bimodule and $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ is an algebra homomorphism with $\|\theta\| \leq 1$, then $\mathcal{A} \times_\theta \mathcal{B}$ is the Banach algebra with the norm $\|(a, b)\|_1 = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}}$.

If we define a norm on $\mathcal{A} \times_\theta \mathcal{B}$ as $|(a, b)| = \max\{\|a\|_{\mathcal{A}} + \|\theta(b)\|_{\mathcal{X}}, \|b\|_{\mathcal{B}}\}$ for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$, then $(\mathcal{A} \times_\theta \mathcal{B}, |\cdot|)$ is also a Banach algebra. In fact, $\|(a, b)\|_1 \leq 2|(a, b)| \leq 2\|(a, b)\|_1$ for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$. If we identify $\mathcal{A} \times \{0\}$ with $\mathcal{A}$ and $\{0\} \times \mathcal{B}$ with $\mathcal{B}$ in $\mathcal{A} \times_\theta \mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are closed ideal and closed subalgebra of $\mathcal{A} \times_\theta \mathcal{B}$ respectively and the quotient $(\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$, i.e., $\mathcal{A} \times_\theta \mathcal{B}$ is a strong splitting Banach algebra extension of $\mathcal{B}$ by $\mathcal{A}$. Throughout the paper, all algebras are considered to be complex algebras.

The above multiplication on $\mathcal{A} \times \mathcal{B}$ generalizes some known multiplication on the product space $\mathcal{A} \times \mathcal{B}$. They are as follows.

(1) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras, let $\mathcal{X} = \mathbb{C}$, and let $\theta : \mathcal{B} \to \mathbb{C}$ be a homomorphism. Then $\mathcal{A}$ is a $\mathbb{C}-$bimodule with respect to the module operations defined as $a \cdot \alpha = \alpha \circ a = \alpha a$ for all $\alpha \in \mathbb{C}$ and $a \in \mathcal{A}$. It can be seen that $\mathcal{A} \times_\theta \mathcal{B}$ is the $\theta-$Lau product of $\mathcal{A}$ and $\mathcal{B}$.

Lau first introduced $\theta-$Lau product in [7] for certain classes of Banach algebras. Later, it was extended and studied by Monfared for general case in [8]. Various Banach algebra properties of $\mathcal{A} \times_\theta \mathcal{B}$ are studied in different papers, for example, [1, 2, 4, 5, 6, 8, 9] etc.

(2) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras, and let $\mathcal{X} = \mathcal{A}$. It is clear that $\mathcal{A}$ is a $\mathcal{A}-$bimodule with respect to the module operations defined as $(a_1, a_2) \in \mathcal{A} \times \mathcal{A} \mapsto a_1a_2 \in \mathcal{A}$ and $(a_1, a_2) \in \mathcal{A} \times \mathcal{A} \mapsto a_2a_1 \in \mathcal{A}$ for all $a_1, a_2 \in \mathcal{A}$. Let $\theta : \mathcal{B} \longrightarrow \mathcal{A}$ be an algebra homomorphism. It can be seen that $\mathcal{A} \times_\theta \mathcal{B}$ is the $T-$Lau product of $\mathcal{A}$ and $\mathcal{B}$ [3].

(3) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras, $\mathcal{B} = \mathcal{X}$, $\mathcal{A}$ be an $\mathcal{X}-$bimodule, and $\theta = I$, the identity map. Then θ is an algebra homomorphism and $\mathcal{A}$ is an algebraic $\mathcal{B}-$module. Then

$\mathcal{A} \times_\theta \mathcal{B}$ is the $\bowtie -$product of $\mathcal{A}$ and $\mathcal{B}$. Ramezanpour and Barootkoob introduced $\bowtie -$product in [10].
(4) Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{X}$ be algebras, and let $\mathcal{A}$ be an $\mathcal{X}-$bimodule. If we define $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ as $\theta(b) = 0$ for all $b \in \mathcal{B}$, then θ is an algebra homomorphism. Clearly, $\mathcal{A} \times_\theta \mathcal{B}$ is the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$.

2. Some basic properties of $\mathcal{A} \times_\theta \mathcal{B}$

An algebra $\mathcal{A}$ is *commutative* if $ab = ba$ for all $a, b \in \mathcal{A}$. An element $e \in \mathcal{A}$ is an *identity* for $\mathcal{A}$ if $ae = a = ea$ for all $a \in \mathcal{A}$.

Lemma 2.1. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$ be algebras, $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ be an algebra homomorphism, and let $\mathcal{A}$ be a symmetric $\mathcal{X}-$bimodule. Then the following statements hold.*

1. *$\mathcal{A} \times_\theta \mathcal{B}$ is commutative if and only if $\mathcal{A}$ and $\mathcal{B}$ are commutative.*
2. *$(0, e_\mathcal{B})$ is the identity for $\mathcal{A} \times_\theta \mathcal{B}$ if and only if $e_\mathcal{B}$ is the identity for $\mathcal{B}$ and $a \cdot \theta(e_\mathcal{B}) = a$ for all $a \in \mathcal{A}$.*

Proof. The statement (1) is a simple verification.

(2) Let $(0, e_\mathcal{B})$ be the identity for $\mathcal{A} \times_\theta \mathcal{B}$. It follows from $(a \cdot \theta(e_\mathcal{B}), be_\mathcal{B}) = (a, b)(0, e_\mathcal{B}) = (a, b) = (0, e_\mathcal{B})(a, b) = (\theta(e_\mathcal{B}) \circ a, e_\mathcal{B} b)$ that $a \cdot \theta(e_\mathcal{B}) = a = \theta(e_\mathcal{B}) \circ a$ and $be_\mathcal{B} = b = e_\mathcal{B} b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Conversely, let $e_\mathcal{B}$ be the identity for $\mathcal{B}$ and $a \cdot \theta(e_\mathcal{B}) = a$ for all $a \in \mathcal{A}$. Since $\mathcal{A}$ is a symmetric $\mathcal{X}-$bimodule, $(a, b)(0, e_\mathcal{B}) = (a \cdot \theta(e_\mathcal{B}), be_\mathcal{B}) = (a, b)$ and $(0, e_\mathcal{B})(a, b) = (\theta(e_\mathcal{B}) \circ a, e_\mathcal{B} b) = (a, b)$ for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$. □

A net $\{e_\alpha\}_{\alpha \in \Lambda}$ of elements of $\mathcal{A}$ is a *bounded left approximate identity* for a normed algebra $(\mathcal{A}, \| \cdot \|_\mathcal{A})$ if there exists some $M > 0$ such that $\|e_\alpha\|_\mathcal{A} \leq M$ for all $\alpha \in \Lambda$ and $\|e_\alpha a - a\|_\mathcal{A} \to 0$ for all $a \in \mathcal{A}$. Similarly, a *bounded right approximate identity* and a *bounded (two sided) approximate identity* are defined.

Proposition 2.2. *Let $(\mathcal{A}, \| \cdot \|_\mathcal{A})$, $(\mathcal{B}, \| \cdot \|_\mathcal{B})$, and $(\mathcal{X}, \| \cdot \|_\mathcal{X})$ be normed algebras, $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ be an algebra homomorphism with $\|\theta\| \leq 1$, and let $\mathcal{A}$ be a normed $\mathcal{X}-$bimodule. Then $\{(e_\alpha, f_\alpha)\}_{\alpha \in \Lambda}$ is a bounded left (right, or two sided) approximate identity for $\mathcal{A} \times_\theta \mathcal{B}$ if and only if $\{f_\alpha\}_{\alpha \in \Lambda}$ is a bounded left (right, or two sided) approximate identity for $\mathcal{B}$, $\{e_\alpha\}_{\alpha \in \Lambda}$ is bounded, $\|e_\alpha a + \theta(f_\alpha) \circ a - a\|_\mathcal{A} \to 0$, and $\|e_\alpha \cdot \theta(b)\|_\mathcal{A} \to 0$.*

Proof. Let $\{(e_\alpha, f_\alpha)\}_{\alpha \in \Lambda}$ be a bounded left approximate identity for $\mathcal{A} \times_\theta \mathcal{B}$. Then there exists some $M > 0$ such that $|(e_\alpha, f_\alpha)| \leq M$ for all $\alpha \in \Lambda$ and $|(e_\alpha, f_\alpha)(a, b) - (a, b)| \to 0$ for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$. By definition of $| \cdot |$, the nets

$\{e_\alpha\}_{\alpha\in\Lambda}$ and $\{f_\alpha\}_{\alpha\in\Lambda}$ are bounded. If $b \in \mathcal{B}$,

$$\begin{aligned}\max\{\|e_\alpha\cdot\theta(b)\|_{\mathcal{A}}, \|f_\alpha b - b\|_{\mathcal{B}}\} &\leq \max\{\|e_\alpha\cdot\theta(b)\|_{\mathcal{A}} + \|\theta(f_\alpha b - b)\|_{\mathcal{X}}, \|f_\alpha b - b\|_{\mathcal{B}}\}\\ &= |(e_\alpha\cdot\theta(b), f_\alpha b - b)| = |(e_\alpha, f_\alpha)(0,b) - (0,b)|.\end{aligned}$$

If $a \in \mathcal{A}$,

$$\begin{aligned}\|e_\alpha a + \theta(f_\alpha)\circ a - a\|_{\mathcal{A}} &= \max\{\|e_\alpha a + \theta(f_\alpha)\circ a - a\|_{\mathcal{A}} + \|0\|_{\mathcal{X}}, \|0\|_{\mathcal{B}}\}\\ &= |(e_\alpha a + \theta(f_\alpha)\circ a - a, 0)| = |(e_\alpha, f_\alpha)(a,0) - (a,0)|.\end{aligned}$$

So, $\|f_\alpha b - b\|_{\mathcal{B}} \to 0$, $\|e_\alpha\cdot\theta(b)\|_{\mathcal{A}} \to 0$, and $\|e_\alpha a + \theta(f_\alpha)\circ a - a\|_{\mathcal{A}} \to 0$.

Assume the converse. Let $(a,b) \in \mathcal{A}\times_\theta\mathcal{B}$. Then

$$\begin{aligned}&|(e_\alpha, f_\alpha)(a,b) - (a,b)|\\ =\ &\max\{\|e_\alpha a + e_\alpha\cdot\theta(b) + \theta(f_\alpha)\circ a - a\|_{\mathcal{A}} + \|\theta(f_\alpha b - b)\|_{\mathcal{X}}, \|f_\alpha b - b\|_{\mathcal{B}}\}\\ \leq\ &\max\{\|e_\alpha a + \theta(f_\alpha)\circ a - a\|_{\mathcal{A}} + \|e_\alpha\cdot\theta(b)\|_{\mathcal{A}} + \|\theta(f_\alpha b - b)\|_{\mathcal{X}}, \|f_\alpha b - b\|_{\mathcal{B}}\}.\end{aligned}$$

It follows from the fact $\|\theta\| \leq 1$ and our assumptions that $\{(e_\alpha, f_\alpha)\}_{\alpha\in\Lambda}$ is a bounded left approximate identity for $\mathcal{A}\times_\theta\mathcal{B}$. □

An element $a \in \mathcal{A}$ is an *idempotent* if $a^2 = a$ and a non-zero idempotent a is a *minimal idempotent* if $a\mathcal{A}a$ is a division algebra or $a\mathcal{A}a = \mathbb{C}a$. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$ be algebras, $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ be an algebra homomorphism. It is clear that $(a,b) \in \mathcal{A}\times_\theta\mathcal{B}$ is an idempotent if and only if $b \in \mathcal{B}$ is an idempotent and $a^2 + a\cdot\theta(b) + \theta(b)\circ a = a$.

Proposition 2.3. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$ be algebras, $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ be an injective algebra homomorphism. Then (a,b) is a minimal idempotent in $\mathcal{A}\times_\theta\mathcal{B}$ if and only if $(a,\theta(b))$ is a minimal idempotent in $\mathcal{A}\bowtie\theta(\mathcal{B})$ and b is a minimal idempotent in $\mathcal{B}$ provided $b \neq 0$.*

Proof. Let $(a,b) \in \mathcal{A}\times_\theta\mathcal{B}$ be a minimal idempotent, i.e., $(a,b)^2 = (a,b)$ and $(a,b)(\mathcal{A}\times_\theta\mathcal{B})(a,b) = \mathbb{C}(a,b)$ or $(a,b)^2 = (a,b)$ and given $(a_0,b_0) \in \mathcal{A}\times_\theta\mathcal{B}$, there exists some $\lambda_{(a_0,b_0)} \in \mathbb{C}$ such that $(a,b)(a_0,b_0)(a,b) = \lambda_{(a_0,b_0)}(a,b)$. So,

$$a^2 + a\cdot\theta(b) + \theta(b)\circ a = a, \tag{1}$$

$$b^2 = b \quad \text{and} \tag{2}$$

$$\begin{aligned}aa_0a + (a\cdot\theta(b_0))a + (\theta(b)\circ a_0)a + aa_0\cdot\theta(b) + (a\cdot\theta(b_0))\cdot\theta(b)&\\ +(\theta(b)\circ a_0)\cdot\theta(b) + \theta(bb_0)\circ a = \lambda_{(a_0,b_0)}a&,\end{aligned} \tag{3}$$

$$bb_0b = \lambda_{(a_0,b_0)}b. \tag{4}$$

It follows from equations (2) and (4) that if $b \neq 0$, then b is a minimal idempotent with $\lambda_{(a_0,b_0)} = \lambda_{(0,b_0)}$ for all $a_0 \in \mathcal{A}$ and it follows from above four equations that $(a,\theta(b))^2 = (a,\theta(b))$ and $(a,\theta(b))(a_0,\theta(b_0))(a,\theta(b)) = \lambda_{(a_0,b_0)}(a,\theta(b))$ for all $(a_0,b_0) \in \mathcal{A}\times_\theta\mathcal{B}$.

Conversely, let $b \in \mathcal{B}$ be a minimal idempotent, i.e., $b^2 = b$ and for given $b_1 \in \mathcal{B}$, there exists some $\lambda_{b_1} \in \mathbb{C}$ such that $bb_1b = \lambda_{b_1}b$. This gives $\theta(bb_1b) = \theta(\lambda_{b_1}b)$. Since $(a, \theta(b)) \in \mathcal{A} \bowtie \theta(\mathcal{B})$ is a minimal idempotent, i.e., $(a, \theta(b))^2 = (a, \theta(b))$ and for given $(a_0, \theta(b_0)) \in \mathcal{A} \bowtie \theta(\mathcal{B})$, there exists some $\lambda_{(a_0,\theta(b_0))} \in \mathbb{C}$ such that $(a, \theta(b))(a_0, \theta(b_0))(a, \theta(b)) = \lambda_{(a_0,\theta(b_0))}(a, \theta(b))$. So, $(a, \theta(b))(a_0, \theta(b_1))(a, \theta(b)) = \lambda_{(a_0,\theta(b_1))}(a, \theta(b))$. It follows from injectivity of θ that $\lambda_{(a_0,\theta(b_1))} = \lambda_{b_1}$. The case $b = 0$ is easy to verify. $\square$

The following example show that the condition that θ is injective in the Proposition 2.3 is necessary.

Example 2.1. We consider the semigroup $\mathbb{N}$ with the gcd binary operation and we denote $\mathbb{N}$ with this binary operation by $\mathbb{N}_{\text{gcd}}$. The semigroup algebra

$$\ell^1(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{C} : \|f\| = \sum_{n\in\mathbb{N}} |f(n)| < \infty\}$$

is a commutative Banach algebra with the above norm and the convolution multiplication

$$(f \star g)(n) = \sum_{\gcd(u,v)=n} f(u)g(v) \quad (f, g \in \ell^1(\mathbb{N}_{\text{gcd}}), n \in \mathbb{N}).$$

We write an element f of $\ell^1(\mathbb{N}_{\text{gcd}})$ by $f = \sum_{n\in\mathbb{N}} f(n)\delta_n$, where $\delta_n : \mathbb{N} \to \mathbb{C}$ is defined by $\delta_n(n) = 1$ and $\delta_n(m) = 0$ if $m \neq n$. Take $\mathcal{A} = \mathcal{B} = \ell^1(\mathbb{N}_{\text{gcd}})$ and define $\theta : \mathcal{B} \to \mathbb{C}$ by $\theta(f) = \sum_{n\in\mathbb{N}} f(2n)$ for all $f \in \mathcal{B}$. Then θ is a complex homomorphism on $\mathcal{B}$ and θ is not injective. Note that $\delta_1 \star \delta_m = \delta_{\gcd(1,m)} = \delta_1$ for all $m \in \mathbb{N}$. So, if $f = \sum_{n\in\mathbb{N}} f(n)\delta_n \in \ell^1(\mathbb{N})$, then $\delta_1 \star f = \sum_{n\in\mathbb{N}} f(n)\delta_1 = \left(\sum_{n\in\mathbb{N}} f(n)\right)\delta_1$. Clearly, $\delta_1 \star \delta_1 = \delta_1$ and $\delta_1 \star f \star \delta_1 = \left(\sum_{n\in\mathbb{N}} f(n)\right)\delta_1$, i.e., δ_1 is a minimal idempotent in $\mathcal{B}$. We now show that $(\delta_1, \theta(\delta_1)) = (\delta_1, 0)$ is a minimal idempotent in $\mathcal{A} \bowtie \theta(\mathcal{B})$. First observe that $(\delta_1, 0)(\delta_1, 0) = (\delta_1, 0)$. Let $(f, \theta(g))$ be in $\mathcal{A} \bowtie \theta(\mathcal{B})$. Then

$$\begin{aligned}
(\delta_1, 0)(f, \theta(g))(\delta_1, 0) &= (\delta_1 \star f + \theta(g)\delta_1, 0)(\delta_1, 0) \\
&= (\left(\sum_{n\in\mathbb{N}} f(n) + \theta(g)\right)\delta_1, 0)(\delta_1, 0) \\
&= (\left(\sum_{n\in\mathbb{N}} f(n) + \theta(g)\right)\delta_1, 0) \\
&= \left(\sum_{n\in\mathbb{N}} f(n) + \theta(g)\right)(\delta_1, 0).
\end{aligned}$$

Therefore $(\delta_1, \theta(\delta_1))$ is a minimal idempotent in $\mathcal{A} \bowtie \theta(\mathcal{B})$. We now show that (δ_1, δ_1) is not a minimal idempotent in $\mathcal{A} \times_\theta \mathcal{B}$. Notice that

$$\begin{aligned}(\delta_1, \delta_1)(\delta_2, \delta_2)(\delta_1, \delta_1) &= (\delta_1 \star \delta_2 + \theta(\delta_2)\delta_1 + \theta(\delta_1)\delta_2, \delta_1)(\delta_1, \delta_1) \\ &= (\delta_1 + \delta_1, \delta_1)(\delta_1, \delta_1) \\ &= (2\delta_1, \delta_1),\end{aligned}$$

and $(2\delta_1, \delta_1) \neq \lambda(\delta_1, \delta_1)$ for any $\lambda \in \mathbb{C}$. Therefore (δ_1, δ_1) is not a minimal idempotent in $\mathcal{A} \times_\theta \mathcal{B}$.

2.1. Ideals of the type $I \times J$ in $\mathcal{A} \times_\theta \mathcal{B}$. A subset I of $\mathcal{A}$ is a *left ideal* in $\mathcal{A}$ if I is a linear subspace of $\mathcal{A}$ and $aI \subseteq I$ for all $a \in \mathcal{A}$. Similarly, a *right ideal* and an *ideal* are defined. A left ideal I is a *modular left ideal in $\mathcal{A}$ with modular unit u* if there exists $u \in \mathcal{A}$ such that $au - a \in I$ for all $a \in \mathcal{A}$. Similarly, a *modular right ideal* and a *modular ideal* are defined. An ideal I is proper if $I \neq \mathcal{A}$. A *proper left ideal* I is *maximal* if $J = I$ or $J = \mathcal{A}$ whenever J is a left ideal in $\mathcal{A}$ containing I. An ideal I is a *prime ideal* if $a \in I$ or $b \in I$ whenever $a, b \in \mathcal{A}$ and $ab \in I$.

Proposition 2.4. *Let K be a left ideal in a Banach algebra $\mathcal{A} \times_\theta \mathcal{B}$. Define two sets $I = \{a \in \mathcal{A} : (a, b) \in K$ for some $b \in \mathcal{B}\}$ and $J = \{b \in \mathcal{B} : (a, b) \in K$ for some $a \in \mathcal{A}\}$. Then the following statements hold.*

(1) *J is a left ideal in $\mathcal{B}$.*
(2) *If θ vanishes on J, then I is a left ideal in $\mathcal{A}$. If in addition $\mathcal{A}$ has a left approximate identity and K is closed in $\mathcal{A} \times_\theta \mathcal{B}$, then $K = I \times J$.*
(3) *If θ does not vanish on J and $\mathcal{A} \cdot \theta(J) \subseteq I$, then I is a left ideal in $\mathcal{A}$*

Proof. (1) Let $b \in J$. Then there exists some $a \in \mathcal{A}$ such that $(a, b) \in K$. Let $b_1 \in \mathcal{B}$. Then $(\theta(b_1) \circ a, b_1 b) = (0, b_1)(a, b) \in K$, i.e., we get an element $\theta(b_1) \circ a \in \mathcal{A}$ such that $(\theta(b_1) \circ a, b_1 b) \in K$. Hence $b_1 b \in J$.

(2) Let θ vanish on J and $a \in I$. Then there exists some $b \in \mathcal{B}$ such that $(a, b) \in K$. Since $a \in I \subseteq \mathcal{A}$, by definition of J, we have $b \in J$. Let $a_1 \in \mathcal{A}$. Then $(a_1 a, 0) = (a_1 a + a_1 \cdot \theta(b), 0) = (a_1, 0)(a, b) \in K$, i.e., we get an element $0 \in \mathcal{B}$ such that $(a_1 a, 0) \in K$. Therefore, $a_1 a \in I$.

Let $\{a_\alpha\}_{\alpha \in \Lambda}$ be a left approximate identity for $\mathcal{A}$ and K be closed in $\mathcal{A} \times_\theta \mathcal{B}$. By definitions of I and J, $K \subseteq I \times J$. Now, let $p \in I$ and $q \in J$. We show that $(p, q) \in K$. Since $p \in I$ and $q \in J$, there exist $b \in \mathcal{B}$ and $a \in \mathcal{A}$ such that $(p, b) \in K$ and $(a, q) \in K$. Then $(a_\alpha a, 0) = (a_\alpha a + a_\alpha \cdot \theta(q), 0) = (a_\alpha, 0)(a, q) \in K$ and $|(a_\alpha a, 0) - (a, 0)| = \|a_\alpha a - a\|_{\mathcal{A}} \to 0$. Since K is closed in $\mathcal{A} \times_\theta \mathcal{B}$, $(a, 0) \in K$. Similarly, we can show that $(p, 0) \in K$. So, $(0, q) = (a, q) - (a, 0) \in K$. Hence $(p, q) = (p, 0) + (0, q) \in K$.

(3) It follows from the proof of (2). □

Lemma 2.5. *Let I and J be two non-empty subsets of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then $I \times J$ is a left ideal in $\mathcal{A} \times_\theta \mathcal{B}$ if and only if I is a left ideal in $\mathcal{A}$, J is a left ideal in $\mathcal{B}$, $\mathcal{A} \cdot \theta(J) \subseteq I$ and $\theta(\mathcal{B}) \circ I \subseteq I$.*

Proof. Let $I \times J$ be a left ideal in $\mathcal{A} \times_\theta \mathcal{B}$. Then for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$ and $(i, j) \in I \times J$, $(ai + a \cdot \theta(j) + \theta(b) \circ i, bj) = (a, b)(i, j) \in I \times J$ or $ai + a \cdot \theta(j) + \theta(b) \circ i \in I$ and $bj \in J$. So, J is a left ideal in $\mathcal{B}$ and $ai + a \cdot \theta(j) + \theta(b) \circ i \in I$. In particular, taking $a = 0$, we get $\theta(b) \circ i \in I$, i.e., $\theta(\mathcal{B}) \circ I \subseteq I$ and taking $i = 0$ and $j = 0$ respectively, we get $a \cdot \theta(j) \in I$, i.e., $\mathcal{A} \cdot \theta(J) \subseteq I$ and $ai + \theta(b) \circ i \in I$ and so $ai = (ai + \theta(b) \circ i) - (\theta(b) \circ i) \in I$.

Assume the converse. Let $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$ and $(i, j) \in I \times J$. Then $(a, b)(i, j) = (ai + a \cdot \theta(j) + \theta(b) \circ i, bj) \in I \times J$ by our assumption. □

Let $\mathcal{A}$ be an $\mathcal{X}-$bimodule. A left ideal I is *a modular left $\mathcal{X}-$ideal in $\mathcal{A}$ with modular $\mathcal{X}-$unit x* if there exists $x \in \mathcal{X}$ such that $ax - a \in I$ for all $a \in \mathcal{A}$. Similarly, a *modular right $\mathcal{X}-$ideal* and a *modular $\mathcal{X}-$ideal* are defined.

Proposition 2.6. *Let I be a left ideal in $\mathcal{A}$ and J be a left ideal in $\mathcal{B}$. Then $I \times J$ is a modular left ideal in $\mathcal{A} \times_\theta \mathcal{B}$ with modular unit (i, j) if and only if I is a modular left $\mathcal{X}-$ideal in $\mathcal{A}$ with modular $\mathcal{X}-$unit $\theta(j)$, J is a modular left ideal in $\mathcal{B}$ with modular unit j, $\mathcal{A} \cdot \theta(J) \subseteq I$ and $\theta(\mathcal{B}) \circ I \subseteq I$.*

Proof. Let $I \times J$ be a modular left ideal in $\mathcal{A} \times_\theta \mathcal{B}$ with modular unit (i, j). Then for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$, $(a, b)(i, j) - (a, b) \in I \times J$ or $ai + a \cdot \theta(j) + \theta(b) \circ i - a \in I$ and $bj - b \in J$. So, J is a modular left ideal in $\mathcal{B}$ with modular unit j and $ai + a \cdot \theta(j) + \theta(b) \circ i - a \in I$. By Lemma 2.5, I is a left ideal in $\mathcal{A}$, J is a left ideal in $\mathcal{B}$, and $\mathcal{A} \cdot \theta(J) \subseteq I$, $\theta(\mathcal{B}) \circ I \subseteq I$. Since I is a left ideal in $\mathcal{A}$, $ai \in I$ and so $a \cdot \theta(j) + \theta(b) \circ i - a \in I$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. In particular, taking $b = 0$, we get $a \cdot \theta(j) - a \in I$ for all $a \in \mathcal{A}$. So, I is a modular left $\mathcal{X}-$ideal in $\mathcal{A}$ with modular $\mathcal{X}-$unit $\theta(j)$.

Assume the converse. Let $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$ and $(i, j) \in I \times J$. Then $(a, b)(i, j) - (a, b) = (ai + a \cdot \theta(j) + \theta(b) \circ i - a, bj - b) \in I \times J$ by our assumptions. □

Lemma 2.7. *Let K be a left ideal in $\mathcal{A} \times_\theta \mathcal{B}$ containing $\{0\} \times \mathcal{B}$. Then there is a left ideal I in $\mathcal{A}$ such that $K = I \times \mathcal{B}$. If K is a left ideal in $\mathcal{A} \times_\theta \mathcal{B}$ containing $\mathcal{A} \times \{0\}$, then there is a left ideal J in $\mathcal{B}$ such that $K = \mathcal{A} \times J$.*

Proof. Let K be a left ideal in $\mathcal{A} \times_\theta \mathcal{B}$ containing $\{0\} \times \mathcal{B}$. Let $I = \{a \in \mathcal{A} : (a, b) \in K \text{ for some } b \in \mathcal{B}\}$. It is enough to prove that I is a left ideal in $\mathcal{A}$. For that, let $i \in I$. Then there exists $b \in \mathcal{B}$ such that $(i, b) \in K$. Since $(0, b) \in \{0\} \times \mathcal{B} \subseteq K$, $(i, 0) \in K$. Let $a \in \mathcal{A}$. Then $(ai, 0) = (a, 0)(i, 0) \in K$. Therefore I is a left ideal in $\mathcal{A}$.

One can prove the second statement in a similar way. □

Proposition 2.8. *Let I and J be left ideals of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then the following statements hold.*

(1) *$I \times \mathcal{B}$ is a maximal left ideal in $\mathcal{A} \times_\theta \mathcal{B}$ if and only if I is a maximal left ideal in $\mathcal{A}$, $\mathcal{A} \cdot \theta(\mathcal{B}) \subseteq I$ and $\theta(\mathcal{B}) \circ I \subseteq I$.*

(2) *$\mathcal{A} \times J$ is a maximal left ideal in $\mathcal{A} \times_\theta \mathcal{B}$ if and only if J is a maximal left ideal in $\mathcal{B}$.*

(3) *$I \times J$ is a maximal left ideal in $\mathcal{A} \times_\theta \mathcal{B}$ if and only if either $I = \mathcal{A}$ with J is a maximal left ideal in $\mathcal{B}$ or $J = \mathcal{B}$ with I is a maximal left ideal in $\mathcal{A}$, $\mathcal{A} \cdot \theta(J) \subseteq I$ and $\theta(\mathcal{B}) \circ I \subseteq I$.*

Proof. It follows from Lemma 2.5 and Lemma 2.7. $\square$

Proposition 2.9. *Let $\mathcal{A}$ be a symmetric $\mathcal{X}-$bimodule, I be an ideal in $\mathcal{A}$, and J be an ideal in $\mathcal{B}$. Then $I \times J$ is a prime ideal in $\mathcal{A} \times_\theta \mathcal{B}$ if and only if I is a prime ideal in $\mathcal{A}$, J is a prime ideal in $\mathcal{B}$, $\mathcal{A} \cdot \theta(J) \subseteq I$ and $\theta(\mathcal{B}) \circ I \subseteq I$.*

Proof. Let $I \times J$ be a prime ideal in $\mathcal{A} \times_\theta \mathcal{B}$. It follows from Lemma 2.5 that I is a left ideal in $\mathcal{A}$, J is a left ideal in $\mathcal{B}$, $\mathcal{A} \cdot \theta(J) \subseteq I$ and $\theta(\mathcal{B}) \circ I \subseteq I$. Let $a_1, a_2 \in \mathcal{A}$ be such that $a_1 a_2 \in I$. Then $(a_1, 0)(a_2, 0) = (a_1 a_2, 0) \in I \times J$. It follows that $a_1 \in I$ or $a_2 \in I$, i.e., I is a prime ideal in $\mathcal{A}$. Similarly, J is a prime ideal in $\mathcal{B}$.

Assume the converse. By Lemma 2.5, $I \times J$ is an ideal in $\mathcal{A} \times_\theta \mathcal{B}$. Let $(a_1, b_1), (a_2, b_2) \in \mathcal{A} \times_\theta \mathcal{B}$ be such that $(a_1, b_1)(a_2, b_2) \in I \times J$ or $a_1 a_2 + a_1 \cdot \theta(b_2) + \theta(b_1) \circ a_2 \in I$ and $b_1 b_2 \in J$. Since J is a prime ideal in $\mathcal{B}$, either $b_1 \in J$ or $b_2 \in J$. We are in a situation of two cases.

Case I: Let $b_1 \in J$. Since $\mathcal{A}$ is a symmetric $\mathcal{X}-$bimodule, $\theta(J) \circ \mathcal{A} = \mathcal{A} \cdot \theta(J) \subseteq I$. So, $\theta(b_1) \circ a_2 = a_2 \cdot \theta(b_1) \in I$. Therefore, $a_1 a_2 + a_1 \cdot \theta(b_2) \in I$. This implies that $(a_1, 0)(a_2, b_2) = (a_1 a_2 + a_1 \cdot \theta(b_2), 0) \in I \times \{0\}$. It is clear that I is a prime ideal in $\mathcal{A}$ if and only if $I \times \{0\}$ is a prime ideal in $\mathcal{A} \times \{0\}$. So, we get either $(a_1, 0) \in I \times \{0\}$ or $(a_2, b_2) \in I \times \{0\}$. If $(a_1, 0) \in I \times \{0\}$ then $a_1 \in I$ and so $(a_1, b_1) \in I \times J$. If $(a_2, b_2) \in I \times \{0\}$ then $a_2 \in \mathcal{I}$ and $b_2 = 0 \in \{0\} \subseteq J$. So, $(a_2, b_2) \in I \times J$.

Case II: Let $b_2 \in J$. Then $a_1 \cdot \theta(b_2) \in I$ and so $a_1 a_2 + \theta(b_1) \circ a_2 \in I$. This implies that $(a_1, b_1)(a_2, 0) = (a_1 a_2 + \theta(b_1) \circ a_2, 0) \in I \times \{0\}$. It follows from the above same argument that either $(a_1, b_1) \in I \times J$ or $(a_2, b_2) \in I \times J$. $\square$

2.2. Gelfand space of $\mathcal{A} \times_\theta \mathcal{B}$. Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{A}^*$ be the dual of $\mathcal{A}$. A nonzero linear map $\varphi : \mathcal{A} \to \mathbb{C}$ is a *complex homomorphism* if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$. Let $\Delta(\mathcal{A})$ be the set of all complex homomorphism on $\mathcal{A}$. Clearly, $\Delta(\mathcal{A}) \subseteq \mathcal{A}^*$. For $a \in \mathcal{A}$, let $\widehat{a} : \Delta(\mathcal{A}) \to \mathbb{C}$ be $\widehat{a}(\varphi) = \varphi(a)$ for all $\varphi \in \Delta(\mathcal{A})$. The weakest topology on $\Delta(\mathcal{A})$ in which all $\widehat{a}$, $a \in \mathcal{A}$, are continuous is the *Gelfand topology* on $\Delta(\mathcal{A})$. The set $\Delta(\mathcal{A})$ with the Gelfand topology is the *Gelfand space* of $\mathcal{A}$. Note that if $a \in \mathcal{A}$, then $\widehat{a} \in C_0(\Delta(\mathcal{A}))$, where $C_0(\Delta(\mathcal{A}))$ is the collection of all continuous functions on $\Delta(\mathcal{A})$ vanishing at infinity. The map

$a \in \mathcal{A} \mapsto \widehat{a} \in C_0(\Delta(\mathcal{A}))$ is the *Gelfand map*. A commutative Banach algebra $\mathcal{A}$ is *semisimple* if the Gelfand map is injective.

Next theorem gives the Gelfand space of $\mathcal{A} \times_\theta \mathcal{B}$.

Theorem 2.10. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$ be commutative Banach algebras, $\mathcal{A}$ be a symmetric Banach $\mathcal{X}-$bimodule, and $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ be an algebra homomorphism with $\|\theta\| \leq 1$. Then the Gelfand space $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ of $\mathcal{A} \times_\theta \mathcal{B}$ is a disjoint union of the sets $E := \{(\varphi, \varphi(a_\varphi \cdot \theta(\cdot))) : \varphi \in \Delta(\mathcal{A}), a_\varphi \in \mathcal{A}$ such that $\varphi(a_\varphi) = 1\}$ and $F := \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}$.*

Proof. Let $\Phi \in \Delta(\mathcal{A}\times_\theta\mathcal{B})$. Since $\Delta(\mathcal{A}\times_\theta\mathcal{B}) \subseteq (\mathcal{A}\times_\theta\mathcal{B})^*$, there exist $\varphi \in \mathcal{A}^*$ and $\psi \in \mathcal{B}^*$ such that $\Phi = (\varphi, \psi)$. Let $(a_1, b_1), (a_2, b_2) \in \mathcal{A}\times_\theta\mathcal{B}$. Then $(\varphi, \psi)[(a_1, b_1)(a_2, b_2)] = (\varphi, \psi)(a_1, b_1)(\varphi, \psi)(a_2, b_2)$ or $(\varphi, \psi)(a_1a_2 + a_1 \cdot \theta(b_2) + \theta(b_1) \circ a_2, b_1b_2) = (\varphi(a_1) + \psi(b_1))(\varphi(a_2) + \psi(b_2))$ or

$$\begin{aligned}\varphi(a_1a_2 + a_1 \cdot \theta(b_2) + \theta(b_1) \circ a_2) + \psi(b_1b_2) &= \varphi(a_1)\varphi(a_2) + \varphi(a_1)\psi(b_2) \\ &\quad + \psi(b_1)\varphi(a_2) + \psi(b_1)\psi(b_2). \end{aligned} \tag{5}$$

In particular, taking $b_1 = b_2 = 0$ and $a_1 = a_2 = 0$ in equation (5) respectively, we get $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ and $\psi(b_1b_2) = \psi(b_1)\psi(b_2)$. The equation (5) gives

$$\varphi(a_1 \cdot \theta(b_2)) + \varphi(\theta(b_1) \circ a_2) = \varphi(a_1)\psi(b_2) + \psi(b_1)\varphi(a_2) \tag{6}$$

for all $(a_1, b_1), (a_2, b_2) \in \mathcal{A} \times_\theta \mathcal{B}$. Let $\varphi \neq 0$. Then there exists $a_\varphi \in \mathcal{A}$ such that $\varphi(a_\varphi) = 1$. Taking $a_1 = a_2 = a_\varphi$ and $b_1 = b_2 = b$ in equation (6), we get $\varphi(a_\varphi \cdot \theta(b)) = \varphi(a_\varphi)\psi(b)$. Therefore, $\psi(\cdot) = \varphi(a_\varphi \cdot \theta(\cdot))$. One can observe that $\psi(\cdot)$ is independent of the choice of a_φ satisfying $\varphi(a_\varphi) = 1$. Indeed, let $a_1, a_2 \in \mathcal{A}$ such that $\varphi(a_1) = 1 = \varphi(a_2)$. Since $\mathcal{A}$ is a symmetric $\mathcal{X}-$bimodule, we have $(a_1 \cdot \theta(\cdot))a_2 = a_1(\theta(\cdot) \circ a_2)$ and so $\varphi((a_1 \cdot \theta(\cdot))a_2) = \varphi(a_1(\theta(\cdot) \circ a_2))$ or $\varphi(a_1 \cdot \theta(\cdot))\varphi(a_2) = \varphi(a_1)\varphi(\theta(\cdot) \circ a_2)$ or $\varphi(a_1 \cdot \theta(\cdot)) = \varphi(\theta(\cdot) \circ a_2)$ for all $\varphi \in \Delta(\mathcal{A})$. Therefore, the map $\varphi(a_\varphi \cdot \theta(\cdot))$ is well-defined. Since $\varphi(a_\varphi) = 1$ and $\mathcal{A}$ is symmetric, the map $\varphi(a_\varphi \cdot \theta(\cdot))$ is multiplicative. Indeed,

$$\begin{aligned}\varphi(a_\varphi \cdot \theta(b_1b_2)) &= \varphi(a_\varphi \cdot (\theta(b_1)\theta(b_2))) = \varphi(a_\varphi \cdot (\theta(b_1)\theta(b_2)))\varphi(a_\varphi) \\ &= \varphi((a_\varphi \cdot (\theta(b_1)\theta(b_2)))a_\varphi) = \varphi([(a_\varphi \cdot \theta(b_1)) \cdot \theta(b_2)]a_\varphi) \\ &= \varphi([a_\varphi \cdot \theta(b_1)][\theta(b_2) \circ a_\varphi]) = \varphi(a_\varphi \cdot \theta(b_1))\varphi(\theta(b_2) \circ a_\varphi) \\ &= \varphi(a_\varphi \cdot \theta(b_1))\varphi(a_\varphi \cdot \theta(b_2))\end{aligned}$$

for all $b_1, b_2 \in \mathcal{B}$. Therefore, $(\varphi, \psi) \in E$. Next, let $\varphi = 0$. Then $\psi \in \Delta(\mathcal{B})$ and $(0, \psi) \in F$. Hence $\Delta(\mathcal{A} \times_\theta \mathcal{B}) \subseteq E \cup F$.

Conversely, let $\Phi \in E \cup F$. Then computation shows that $\Phi \in \Delta(\mathcal{A} \times_\theta \mathcal{B})$. $\square$

Corollary 2.11. *Assume the hypothesis of Theorem 2.10. Then sets E and F are open and closed in $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ respectively.*

PROOF. Let $(\varphi, \varphi(\theta(\cdot) \circ a_\varphi)) \in E$. Since $\varphi \in \Delta(\mathcal{A})$, there exists $a_0 \in \mathcal{A}$ such that $\varphi(a_0) \neq 0$. Take $\varepsilon = \frac{|\varphi(a_0)|}{4}$ and $U = U\left((\varphi, \varphi(\theta(\cdot) \circ a_\varphi)), (a_0, 0), \varepsilon\right)$. Then U is a neighborhood of $(\varphi, \varphi(\theta(\cdot) \circ a_\varphi))$ and

$$\begin{aligned} U &= \{(\varphi_1, \psi_1) \in \Delta(\mathcal{A} \times_\theta \mathcal{B}) : |(\varphi_1, \psi_1)((a_0, 0)) - (\varphi, \varphi(\theta(\cdot) \circ a_\varphi))\,((a_0, 0))| < \varepsilon\} \\ &= \{(\varphi_1, \psi_1) \in \Delta(\mathcal{A} \times_\theta \mathcal{B}) : |\varphi_1(a_0) - \varphi(a_0)| < \varepsilon\}\,. \end{aligned}$$

If we take any point $(0, \psi) \in U$, then $4\varepsilon = |\varphi(a_0)| < \varepsilon$, a contradiction. Therefore, $U \subset E$. Hence E is open in $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ and so F is closed in $\Delta(\mathcal{A} \times_\theta \mathcal{B})$. $\square$

Corollary 2.12. *Assume the hypothesis of Theorem 2.10. Then $\mathcal{A} \times_\theta \mathcal{B}$ is semisimple if and only if both $\mathcal{A}$ and $\mathcal{B}$ are semisimple.*

PROOF. Let $\mathcal{A} \times_\theta \mathcal{B}$ be semisimple. Let $a \in \mathcal{A}$ and $b \in \mathcal{B}$ satisfy $\widehat{a} = 0$ and $\widehat{b} = 0$, i.e., $\widehat{a}(\varphi) = 0$ and $\widehat{b}(\psi) = 0$ for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. Then $\widehat{(a, 0)}(\Phi) = 0$ and $\widehat{(0, b)}(\Phi) = 0$ for all $\Phi \in \Delta(\mathcal{A} \times_\theta \mathcal{B})$. Since $\mathcal{A} \times_\theta \mathcal{B}$ is semisimple, $a = b = 0$.

Conversely, let $\mathcal{A}$ and $\mathcal{B}$ be semisimple. Let $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$ be such that $\widehat{(a, b)}((\varphi, \psi)) = 0$ for all $(\varphi, \psi) \in \Delta(\mathcal{A} \times_\theta \mathcal{B})$. In particular, taking $\varphi = 0$, we get $\widehat{b}(\psi) = \widehat{(a, b)}((0, \psi)) = 0$. It follows from semisimplicity of $\mathcal{B}$ that $b = 0$, which implies that $\widehat{a}(\varphi) = \widehat{(a, 0)}((\varphi, \varphi(\theta(0) \circ a)) = 0$. The semisimplicity of $\mathcal{A}$ implies that $a = 0$. $\square$

2.3. Module multipliers of $\mathcal{A} \times_\theta \mathcal{B}$. Let $\mathcal{A}$, X, and $\mathcal{X}$ be Banach algebras, and let $\mathcal{A}$ be an $\mathcal{X}-$bimodule. Then X is a *Banach $\mathcal{A} - \mathcal{X}-$bimodule* if X is a Banach $A-$bimodule as well as a Banach $\mathcal{X}-$bimodule which satisfies conditions $(ax)\alpha = a(x\alpha)$, $x(\alpha a) = (x\alpha)a$, $\alpha(ax) = (\alpha a)x$, $\alpha(xa) = (\alpha x)a$, $x(a\alpha) = (xa)\alpha$, and $(a\alpha)x = a(\alpha x)$ for all $a \in \mathcal{A}$, $x \in X$, and $\alpha \in \mathcal{X}$. Let X be an $\mathcal{A}-$bimodule, and let $Ann_X(\mathcal{A}) := \{x \in X : ax = 0 = xa \text{ for all } a \in \mathcal{A}\}$ be the *annihilator* of $\mathcal{A}$ in X. A homomorphism $T : \mathcal{A} \to X$ is a *module homomorphism* if $T(a_1a_2) = T(a_1)a_2$ and $T(a_1a_2) = a_1T(a_2)$ for all $a_1, a_2 \in \mathcal{A}$. Moreover, if $T(a_1)a_2 = a_1T(a_2)$ for all $a_1, a_2 \in \mathcal{A}$, then it is a *module multiplier*. Let $M(\mathcal{A}, X)$ be the set of all module multipliers from $\mathcal{A}$ to X.

Let X be a Banach $\mathcal{A} - \mathcal{X}-$bimodule, Y be a Banach $\mathcal{B} - \mathcal{X}-$bimodule, and $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ be a module homomorphism. Define *module multiplications* on $X \times Y$ as $\alpha(x, y) = (\alpha x, \alpha y)$, $(x, y)\alpha = (x\alpha, y\alpha)$, $(a, b)(x, y) = (ax + \theta(b)x, by)$, and $(x, y)(a, b) = (xa + x\theta(b), yb)$, respectively, for all $\alpha \in \mathcal{X}, (x, y) \in X \times Y$ and $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$.

Lemma 2.13. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$ be Banach algebras, $\mathcal{A}$ be a symmetric Banach $\mathcal{X}-$bimodule, $\mathcal{B}$ be a Banach $\mathcal{X}-$bimodule, and let $\theta : \mathcal{B} \longrightarrow \mathcal{X}$ be a module homomorphism with $\|\theta\| \leq 1$. If X is a Banach $\mathcal{A} - \mathcal{X}-$bimodule and Y is a Banach $\mathcal{B} - \mathcal{X}-$bimodule, then $X \times Y$ is a Banach $(\mathcal{A} \times_\theta \mathcal{B}) - \mathcal{X}-$bimodule with the above module multiplications.*

Proof. We have X a Banach $\mathcal{A}-\mathcal{X}-$bimodule and Y a Banach $\mathcal{B}-\mathcal{X}-$bimodule with the module multiplication $(a,x)\in\mathcal{A}\times X\mapsto ax\in X$, $(x,a)\in X\times\mathcal{A}\mapsto xa\in X$, $(\alpha,x)\in\mathcal{X}\times X\mapsto\alpha x\in X$, $(x,\alpha)\in X\times\mathcal{X}\mapsto x\alpha\in X$, $(b,y)\in\mathcal{B}\times Y\mapsto by\in Y$, $(y,b)\in Y\times\mathcal{B}\mapsto yb\in Y$, $(\alpha,y)\in\mathcal{X}\times Y\mapsto\alpha y\in Y$, and $(y,\alpha)\in Y\times\mathcal{X}\mapsto y\alpha\in Y$ for all $\alpha\in\mathcal{X}$, $x\in X$, $y\in Y$, $a\in\mathcal{A}$, and $b\in\mathcal{B}$. Define four mappings as below.

(1) $(\alpha,(x,y))\in\mathcal{X}\times(X\times Y)\mapsto\alpha(x,y)\in(X\times Y)$,
(2) $((x,y),\alpha)\in(X\times Y)\times\mathcal{X}\mapsto(x,y)\alpha\in(X\times Y)$,
(3) $((a,b),(x,y))\in(\mathcal{A}\times_\theta\mathcal{B})\times(X\times Y)\mapsto(a,b)(x,y)\in(X\times Y)$, and
(4) $((x,y),(a,b))\in(X\times Y)\times(\mathcal{A}\times_\theta\mathcal{B})\mapsto(x,y)(a,b)\in(X\times Y)$,

where module multiplications are defined as said in hypothesis. One may verify that $X\times Y$ together with above module multiplications satisfies all conditions to be $(\mathcal{A}\times_\theta\mathcal{B})-\mathcal{X}-$bimodule. □

The following theorem gives characterization of module multipliers from $\mathcal{A}\times_\theta\mathcal{B}$ to $X\times Y$.

Theorem 2.14. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$ be algebras, $\mathcal{A}$ be an $\mathcal{X}-$bimodule, $\mathcal{B}$ be an $\mathcal{X}-$bimodule, $\theta:\mathcal{B}\longrightarrow\mathcal{X}$ be a module homomorphism, X be a symmetric $\mathcal{A}$-$\mathcal{X}$-bimodule, Y be a $\mathcal{B}$-$\mathcal{X}$-bimodule with $Ann_Y(\mathcal{B})=\{0\}$, and let $T:\mathcal{A}\times_\theta\mathcal{B}\longrightarrow X\times Y$ be a module homomorphism. Then $T\in M(\mathcal{A}\times_\theta\mathcal{B},X\times Y)$ if and only if there exists module homomorphisms $T_1:\mathcal{A}\times_\theta\mathcal{B}\longrightarrow X$ and $T_2:\mathcal{A}\times_\theta\mathcal{B}\longrightarrow Y$ such that $T=(T_1,T_2)$, $T_1|_{\mathcal{A}\times\{0\}}\in M(\mathcal{A},X)$, $T_2|_{\mathcal{A}\times\{0\}}=\{0\}$, $T_2|_{\{0\}\times\mathcal{B}}\in M(\mathcal{B},Y)$, and $\theta(b_1)T_1(a_2,b_2)=T_1(0,b_1)a_2+T_1(0,b_1)\theta(b_2)$ for all $a_2\in\mathcal{A}$ and $b_1,b_2\in\mathcal{B}$.*

Proof. Let $T\in M(\mathcal{A}\times_\theta\mathcal{B},X\times Y)$. Let $(a_1,b_1),(a_2,b_2)\in\mathcal{A}\times_\theta\mathcal{B}$. Then $(T(a_1,b_1))(a_2,b_2)$
$=(a_1,b_1)(T(a_2,b_2))$ or $(T_1(a_1,b_1),T_2(a_1,b_1))(a_2,b_2)=(a_1,b_1)(T_1(a_2,b_2),T_2(a_2,b_2))$ or

$$T_1(a_1,b_1)a_2+T_1(a_1,b_1)\theta(b_2)=a_1T_1(a_2,b_2)+\theta(b_1)T_1(a_2,b_2)\tag{7}$$

and

$$T_2(a_1,b_1)b_2=b_1T_2(a_2,b_2).\tag{8}$$

Taking $a_1=b_2=0$ in equations (7) and (8), we get $T_1(0,b_1)a_2=\theta(b_1)T_1(a_2,0)$ and $b_1T_2(a_2,0)=0$. Since $\mathrm{Ann}_Y(\mathcal{B})=\{0\}$, $T_2(a_2,0)=0$ for all $a_2\in\mathcal{A}$, i.e., $T_2|_{\mathcal{A}\times\{0\}}=\{0\}$. Taking $b_1=b_2=0$ in equations (7) and (8), we get $T_1(a_1,0)a_2=a_1T_1(a_2,0)$ for all $a_1,a_2\in\mathcal{A}$, i.e., $T_1|_{\mathcal{A}\times\{0\}}\in M(\mathcal{A},X)$. Taking $a_1=a_2=0$ in equations (7) and (8), we get $T_1(0,b_1)\theta(b_2)=\theta(b_1)T_1(0,b_2)$ and $T_2(0,b_1)b_2=b_1T_2(0,b_2)$ for all $b_1,b_2\in\mathcal{B}$, i.e., $T_2|_{\{0\}\times\mathcal{B}}\in M(\mathcal{B},Y)$. One may observe that for all $a_2\in\mathcal{A}$ and $b_1,b_2\in\mathcal{B}$,

$$\begin{aligned}\theta(b_1)T_1(a_2,b_2)&=\theta(b_1)[T_1(a_2,0)+T_1(0,b_2)]\\&=\theta(b_1)T_1(a_2,0)+\theta(b_1)T_1(0,b_2)\\&=T_1(0,b_1)a_2+T_1(0,b_1)\theta(b_2).\end{aligned}$$

The converse can be verified easily. □

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