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Extension of the double Newton's method convergence order via the bi-variate power series weight function for solving nonlinear models

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Abstract. This manuscript put forward one and two-parameter families of modified double Newton iterative structure with convergence order six, for approximation of the solution of nonlinear model. The modification technique involves the introduction of quotient of two converging bi-variate Power series based weight function to the second step of the double Newton's method. Some particular members of the developed methods have experimented on some physical phenomena modeled into nonlinear equations and results compared with some existing methods.

1. Introduction

Several real phenomena have been and are continuously modeled into nonlinear model (NLM) of the form $\psi(x) = 0$, and for better insight into the model, its solution δ is often required. Unfortunately, there is no existing unified analytic structure for obtaining the solution the NLM, hence iterative structures are resorted. Since the emergence of the classical convergence order (CO) two Newton's method (NM) [14] put forward as:

$$x_{k+1} = x_k - \frac{\psi(x_k)}{\psi'(x_k)}; \tag{1}$$

modification have been made on it with the aim of improving its CO and efficiency. The use of the composition, weight function or both techniques have been exploited

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by many authors with the sole aim of modifying (1) to attaining higher CO and EI. For example, an early modification to iterative structure (1) is the double Newton method (DNM) [14] designed by composing two NM that yielded a corresponding iterative structure as:

$$\begin{aligned} y_k &= x_k - \frac{\psi(x_k)}{\psi'(x_k)}; \\ x_{k+1} &= y_k - \frac{\psi(y_k)}{\psi'(y_k)}. \end{aligned} \tag{2}$$

Although the iterative structure (2) is of convergence order (CO) 4 and can be considered as an improvement of (1), the efficiency index EI remains 1.4142. The n-times composition of the NM will produce higher CO iterative structure with no changes in EI because, more functions evaluation will be required in an iterative cycle as n increases . In [7], Ogbereyivwe and Muka noted that the golden principles for developing new iterative structure for solving NLM is that the method should attain high CO by utilizing few numbers of functions evaluations and be simply structured. Consequently, many authors had this rule in mind, in putting forward new iterative structures with better CO and EI via the application of the composition and weight function(s) techniques. For instance, in the two sets of works ([1, 2, 3, 5, 8, 9, 10, 12]) and some reference therein, two and three step composition of the NM and many types of functions of iterative structure(s) with weight function(s) were employed to present several CO four and six iterative structures respectively with EI higher that that of (1) and (2). In Ghanbari[3], the structure of the weight function used in the second step of the DNM is a quotient of two, one-variate power series of order two. Further, Lee and Kim in [6] used certain order two, bi-variate power series as weight function in the second step of the DNM (2).

As a follow up to these research trends, a quotient of two kinds of bi-variate power series and their variants are utilised as weight functions attached to the second step of iterative structure (2) to develop CO six iterative structures with better EI than that of (2) for solving NLM. The remaining parts of this manuscript includes the main contributions of this work presented in Section 2, the developed methods implementation on some test problems and comparison are provided in Section 3 and conclusion given in Section 4.

2. Methods Formation

The main contributions of this manuscript is presented in the two subsections of this section. The first subsection presents the modified DNM developed via the use of quotient of two second order bi-variate power series as weight function in its second step, while in the second subsection, the variant of the weight function is utilised.

2.1. The First Family of Power Series Based DNM. In this subsection, a new family of an iterative structure is constructed by the introduction of the quotient of two second order convergent bi-variate power series $G(s,u)$, in the second step of the DNM. Consequently, the corresponding iterative structure is put forward as:

$$\begin{cases} y_k = x_k - \dfrac{\psi(x_k)}{\psi'(x_k)}; \\ x_{k+1} = y_k - \dfrac{\psi(y_k)}{\psi'(y_k)} G(s,u); \\ G(s,u) = \left(1 + \displaystyle\sum_{i=1}^{2} (a_i + a_{i+1}s)\, u^i\right) / \left(1 + \displaystyle\sum_{i=1}^{2} (b_i + b_{i+1}s)\, u^i\right), \end{cases} \tag{3}$$

where $s = \frac{\psi'(y)}{\psi'(x)}$, $u = \frac{\psi(y)}{\psi(x)}$ and $a_i, b_i, \{i = 1, 2, 3\}$ are real parameters to be determined and are responsible for ensuring the convergence of the method, with high order and precision. To determine the convergence of IM (3), it is required to obtain its Asymptotic error equation in the form $\xi_{i+1} = \eta\xi_i^{\rho} + O(\xi_i^{\rho+1})$, (where $\xi_i = x_i - \delta$ is the method's error at ith iteration point i, and δ is the exact solution of $\psi(x) = 0$), via the Taylor series expansion of $\psi(\cdot)$ and $\psi'(\cdot)$ as contained in the iterative structure. When the error equation is obtained, the quantities ρ and η are referred to as the method's CO and asymptotic error constant respectively. Further, the EI of the method is computed as $\rho^{\frac{1}{\tau}}$ (where τ is the number different functions evaluation in (3)).

The proof of the next theorem, establishes the convergence of the method (3).

Theorem 2.1. *Suppose the scalar function $\psi : D \subset R \to R$ has a simple solution δ and is differentiable for at least four times in D and $\psi'(\cdot) \neq 0$. Again, for a choice of x_0 close to δ, the sequence of approximation $\{x_i\}_{i\geq 0}, (x_j \in D)$, produced by the family of IM in (3) converges to δ with CO six when the conditions on the parameters a_i and b_i holds as following: $a_2 = -a_1 - 2, a_3 = -a_1 - 7, b_1 = a_1 - 2, b_2 = -a_1$ and $b_3 = -a_1 - 6$.*

PROOF. By the replacement of x with x_i in the Taylor series of $\psi(x)$ and $\psi'(x)$ about δ, the following expressions are obtained:

$$\psi(x_i) = \psi'(\delta)\left(\xi_1 + \sum_{n=2}^{4} c_n \xi_i^n + O\left(\xi_i^5\right)\right), \tag{4}$$

and

$$\psi'(x_i) = \psi'(\delta)\left(1 + 2c_2\xi_k + 3c_3\xi_k^2 + \cdots + 7c_7\xi_k^6 + O\left(\xi_i^7\right)\right), \tag{5}$$

where $c_j = \frac{1}{j!}\frac{\psi^{(j)}(\delta)}{\psi'(\delta)}$, $j \geq 2$.

When the expressions in (4) and (5) are substituted in the first step of (3), the series expansion for y is obtained as:

$$\begin{aligned} y_k =& \delta + c_2\xi_k^2 + \left(2c_3 - 2c_2^2\right)\xi_k^3 + \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\xi_k^4 \\ &+ \left(4c_5 - 10c_2c_4 - 6c_3^2 + 20c_2^2c_3 - 8c_2^8\right)\xi_k^5 \\ &+ \left(5c_6 - 13c_2c_5 - 17c_3c_4 + 28c_2^2c_4 + 33c_2c_3^2 - 52c_2^3c_3 + 16c_2^5\right)\xi_k^6 + O\left(\xi_k^7\right) \end{aligned} \tag{6}$$

Again, using (4) and (5) with the Taylor expansions of $\psi(y)$ and $\psi'(y)$, the corresponding expansions for u_k and s_k are obtained respectively as:

$$\begin{aligned} u_k = & c_2\xi_k + \left(2c_3 - 2c_2^2\right)\xi_2^2 + \left(3c_4 - 10c_2c_3 + 8c_2^3\right)\xi_k^3 \\ &+ \left(4c_5 - 14c_2c_4 - 8c_3^2 + 37c_2^2 - 20c_2^4\right)\xi_k^4 \\ &+ \left(5c_6 - 18c_2c_5 - 22c_3c_4 + 51c_2^2c_4 + 55c_2c_3^3 - 118c_2^3 + 48c_2^5\right)\xi_k^6 + O\left(\xi_k^7\right) \end{aligned} \tag{7}$$

and

$$\begin{aligned} s_k = & 1 + 2c_2\xi_k + \left(-3c_3 + 6c_2^2\right)\xi_k^2 - 4\left(c_4 - 4c_2c_3 + 4c_2^3\right)\xi_k^3 \\ &+ \left(-5c_5 - 22c_2c_4 + 9c_3^2 - 61c_2^2 + 40c_2^4\right)\xi_k^4 \\ &+ \left(-6c_6 + 28c_2c_5 + 24c_3c_4 - 88c_2^2c_4 - 66c_2c_3^2 + 198c_2^3c_3 - 96c_2^5\right)\xi_k^5 \\ &+ \left(-7c_7 + c_2\right)\left(34c_6 - 194c_3c_4\right) + 30c_3c_5 + 7c_2^2\left(-16c_5 + 415c_3^2\right) \\ &+ 16c_4^2 + 300c_2^3 - 15c_3^3 - 584c_2^4c_3 + 224c_2^6)\xi_k^6 + O\left(\xi_k^7\right). \end{aligned} \tag{8}$$

Now;

$$\begin{aligned} \frac{\psi\left(y_k\right)}{\psi'\left(y_k\right)} =& c_2\xi_k + \left(2c_3 - 2c_2^2\right)\xi_k^3 + \left(3c_4 - 7c_2c_3 + 3c_2^3\right)\xi_k^4 \\ &- 2\left(-2c_5 + 5c_2c_4 + 3c_3^2 - 8c_2^2 + 2c_2^4\right)\xi_k^5 \\ &+ \left(5c_6 - 13c_2c_5 - 17c_3c_4 + 22c_2^2c_4 + 29c_2c_3^2 - 32c_2^3c_3 + 6c_2^5\right)\xi_k^5 + O\left(\xi_k^6\right). \end{aligned} \tag{9}$$

From the expressions in (7) and (8),

$$\begin{aligned} 1+&\sum_{i=1}^{2}\left(a_i + a_{i+1}s\right)u^i = 1 + \left(a_1 + a_2\right)c_2\xi_k \\ &+ \left(-3a_1c_2^2 - 4a_2c_2^2 + a_3c_2^2 + 2a_1c_3 + 2a_2c_3\right)\xi_k^2 \\ &+ \left(4a_3c_2\left(c_3 - 2c_2^2\right) + a_2\left(14c_2^3 - 13c_2c_3 + 3c_4\right) + a_1\left(8c_2^3 - 10c_2c_3 + 3c_4\right)\right)\xi_k^3 \\ &+ \left(a_3\left(43c_2^4 - 43c_2^2c_3 + 4c_3^2 + 6c_2c_4\right) + a_2\left(-45c_2c_2^4 + 62c_2^2 - 18c_2c_4 + 4c_5\right)\right) \\ &+ a_1\left(-20c_2^4 + 37c_2^2c_3 - 8c_3^2 - 14c_2c_4 + 4c_5\right))\xi_k^4 \\ &+ \left(2a_3\left(-95c_2^5 + 144c_2^3c_3 - 38c_2c_3^2 - 31c_2^2c_4 + 6c_3c_4 + 4c_2c_5\right)\right) \\ &+ a_2\left(136c_2^5 - 251c_2^3c_3 + \cdots + 5c_6\right) + a_1\left(48c_2^5 - 118c_2^3c_3 + \cdots + 5c_6\right))\xi_k^5 + O\left(\xi_k^6\right) \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
1+\sum_{i=1}^{2}\left(b_i+b_{i+1}s\right)u^i &= 1+\left(b_1+b_2\right)c_2\xi_k \\
&+\left(-3b_1c_2^2-4b_2c_2^2+b_3c_2^2+2b_1c_3+2b_2c_3\right)\xi_k^2 \\
&+\left(4b_3c_2\left(c_3-2c_2^2\right)+b_2\left(14c_2^3-13c_2c_3+3c_4\right)+b_1\left(8c_2^3-10c_2c_3+3c_4\right)\right)\xi_k^3 \\
&+\left(b_3\left(43c_2^4-43c_2^2c_3+4c_3^2+6c_2c_4\right)+b_2\left(-45c_2c_2^4+62c_2^2-18c_2c_4+4c_5\right)\right) \\
&+b_1\left(-20c_2^4+37c_2^2c_3-8c_3^2-14c_2c_4+4c_5\right))\xi_k^4 \\
&+\left(2b_3\left(-95c_2^5+144c_2^3c_3-38c_2c_3^2-31c_2^2c_4+6c_3c_4+4c_2c_5\right)\right) \\
&+b_2\left(136c_2^5-251c_2^3c_3+\cdots+5c_6\right)+b_1\left(48c_2^5-118c_2^3c_3+\cdots+5c_6\right))\xi_k^5+O\left(\xi_k^6\right).
\end{aligned} \tag{11}
$$

The quotient of (10) and (11) is:

$$
\begin{aligned}
G\left(s,u\right)=1&+\left(a_1+a_2-b_1-b_2\right)c_2\xi_k+\left(a_3c_2^2+3b_1c_2^2+b_1^2c_2^2+4b_2c_2^2+2b_1b_2c_2^2+b_2^2\right) \\
&-b_3c_2^2-a_1\left(\left(3+b_1+b_2\right)c_2^2-2c_3\right)-a_2\left(\left(4+b_1+b_2\right)c_2^2-2c_3\right) \\
&-2b_1c_3-2b_2b_3)\xi_k^3+\sum_{i=3}^{6}\Omega_i\xi_k^i+O\left(\xi_k^7\right).
\end{aligned} \tag{12}
$$

Using (6), (9) and (12) in the second step of (3), results to the error equation:

$$
\begin{aligned}
x_{k+1}=&\delta-\left(\left(a_1+a_2-b_1-b_2\right)c_2^2\xi_k^3\right) \\
&-c_2\left(\left(1-a_3-5b_1-b_1^2-6b_2-2b_1b_2-b_2^2+a_1\left(5+b_1+b_2\right)\right)\right) \\
&+a_2\left(\left(6+b_1+b_2\right)+b_3\right)c_2^2+4\left(\left(-a_1-a_2+b_1+b_2\right)c_3\right))\xi_k^4 \\
&+\sum_{i=5}^{6}\eta_i\xi_k^i+O\left(\xi_k^7\right),
\end{aligned} \tag{13}
$$

where Ω_i and η_i are multi-variate polynomials that depends on the parameters $a_i, b_i\,(i=1,2,3)$ and $c_j\,(j=2,3,4,5)$. For the error equation in (13) to be of order 6, the coefficients of ξ_k^i, $i=3,4,5,6$ must be annihilated. By equating the coefficients to zero and solve for the parameters yields: $a_2=-2-a_1, a_3=-7-a_1, b_1=-2+a_1, b_2=-a_1$ and $b_3=-6-a_1$. Consequently, the error equation in (13) reduces to:

$$
x_{k+1}=\delta+c_2^2\left(-6c_2c_3+c_4\right)\xi_k^6+O\left(\xi_k^7\right). \tag{14}
$$

The error equation (14) implies that the CO of the modified DNM in (3) is 6. □

Remark 2.1. When $a_2=-2-a_1, a_3=-7-a_1, b_1=-2+a_1, b_2=-a_1$ and $b_3=-6-a_1$ in the IM (3), its convergence is guaranteed and the corresponding iterative structure becomes a one-parameter family of the form:

$$x_{k+1}=x_k-\frac{\psi\left(x_k\right)}{\psi'\left(x_k\right)}-\frac{\psi'\left(y_k\right)}{\psi'\left(x_k\right)}G\left(s,u\right); \tag{15}$$

$$G\left(s,u\right)=\frac{1+\left[a_1-\left(2+a_1\right)\frac{\psi'(y_k)}{\psi'(x_k)}\right]\frac{\psi(y_k)}{\psi(x_k)}-\left[\left(2+a_1\right)+\left(7+a_1\right)\frac{\psi'(y_k)}{\psi'(x_k)}\right]\left(\frac{\psi(y_k)}{\psi(x_k)}\right)^2}{1+\left[\left(a_1-2\right)-a_1\frac{\psi'(y_k)}{\psi'(x_k)}\right]\frac{\psi(y_k)}{\psi(x_k)}-\left[a_1+\left(6+a_1\right)\frac{\psi'(y_k)}{\psi'(x_k)}\right]\left(\frac{\psi(y_k)}{\psi(x_k)}\right)^2} \tag{16}$$

Since (15) has CO six requiring evaluation of four distinct functions in one complete iteration cycle, for any concrete member of it will have EI of 1.5651. This is higher than the EI of the DNM (2).

Remark 2.2. For a concrete member of (15), a_1 is assigned any real value in R. For instance, if $a_1=0$ the IM denoted as M_1 is obtained as:

$$x_{k+1}=x_k-\frac{\psi\left(x_k\right)}{\psi'\left(x_k\right)}-\frac{\psi'\left(y_k\right)}{\psi'\left(x_k\right)}\left(\frac{1-2\frac{\psi'(y_k)}{\psi'(x_k)}\frac{\psi(y_k)}{\psi(x_k)}-\left(2+7\frac{\psi'(y_k)}{\psi'(x_k)}\right)\left(\frac{\psi(y_k)}{\psi(x_k)}\right)^2}{1-2\frac{\psi'(y_k)}{\psi'(x_k)}\frac{\psi(y_k)}{\psi(x_k)}-6\frac{\psi'(y_k)}{\psi'(x_k)}\left(\frac{\psi(y_k)}{\psi(x_k)}\right)^2}\right). \tag{17}$$

2.2. The Second Family of Power series Based DNM. In this subsection, a new family of an iterative structure is constructed by replacing the second order convergent bi-variate power series weight function used in (3) with its variant as following:

$$\begin{cases} y_k=x_k-\dfrac{\psi\left(x_k\right)}{\psi'\left(x_k\right)};\\ x_{k+1}=y_k-\dfrac{\psi'\left(y_k\right)}{\psi'\left(y_k\right)}H\left(s,u\right);\\ H\left(s,u\right)=\left(1+\displaystyle\sum_{i=0}^{1}\left(a_{2i+1}+a_{2i+2}s\right)u^{i+1}\right)/\left(1+\displaystyle\sum_{i=0}^{1}\left(b_{2i}+b_{2i+2}s\right)u^{i+1}\right). \end{cases} \tag{18}$$

The main objective here, is to determine the parameters a_i and b_i, $\{i=1,2,3,4\}$ so as the method (17) estimates the solution of NLM with CO six. To achieve this, the proof of the following theorem is required.

Theorem 2.2. *Suppose the scalar function $\psi:D\subset R\rightarrow R$ has a simple solution δ and is differentiable for at least four times in D and $\psi'(\cdot)\neq 0$. Again, for a choice of x_0 close to δ, the sequence of approximation $\{x_i\}_{i\geq 0}, (x_j\in D)$, produced by the family of IM in (17) converges to δ with CO six when the conditions on the parameters a_i and b_i holds as following: $a_2=-6-a_1, a_4=-a_3-2a_1-1, b_1=a_1-2, b_2=-4-a_1$,$b_3=4+a_3$ and $b_4=-2-a_3-2a_1$.*

PROOF. From the Taylor series expansions in (4)-(9), the following is obtained:

$$
\begin{aligned}
1+\sum_{i=0}^{1}\left(a_{2i+1}+a_{2i+2}s\right)u^{i+1} = {}& 1+\left(a_1+a_2\right)c_2\xi_k \\
&+\left(-2a_2c_2^2+\left(a_3+a_4\right)c_2^2+\left(a_1+a_2\right)\left(-3c_2^2+2c_3\right)\right)\xi_k^2 \\
&+\left(-6a_3c_2^3-8a_4c_2^3+4a_3c_2c_3+4a_3c_2c_3+4a_4c_2c_3\right. \\
&\left.+a_2\left(20c_2^3-17c_2c_3+3c_4+a_1\left(8c_2^3-10c_2c_3+3c_4\right)\right)\right)\xi_k^3 \\
&+\sum\Psi_1\xi_{kk}^4+\sum\Psi_2\xi_{kk}^6+O\left(\xi_k^7\right)
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
1+\sum_{i=0}^{1}\left(b_{2i+1}+b_{2i+2}s\right)u^{i+1} = {}& 1+\left(b_1+b_2\right)c_2\xi_k \\
&+\left(-2b_2c_2^2+\left(b_3+b_4\right)c_2^2+\left(b_1+b_2\right)\left(-3c_2^2+2c_3\right)\right)\xi_k^2 \\
&+\left(-6b_3c_2^3-8b_4c_2^3+4b_3c_2c_3+4b_3c_2c_3+4b_4c_2c_3\right. \\
&\left.+b_2\left(20c_2^3-17c_2c_3+3c_4+b_1\left(8c_2^3-10c_2c_3+3c_4\right)\right)\right)\xi_k^3 \\
&+\sum\Psi_3\xi_k^4+\sum\Psi_4\xi_k^6+O\left(\xi_k^7\right).
\end{aligned}
\tag{20}
$$

The quotient of (18) and (19) is:

$$
\begin{aligned}
H\left(s,u\right) = 1+{}&\left(a_1+a_2-b_1-b_2\right)c_2\xi_k+\left(-2a_2c_2^2+\left(a_3+a_4\right)c_2^2+3b_1c_2^2+b_1^2c_2^2\right. \\
&+5b_2c_2^2+2b_1b_2c_2^2+b_2^2c_2^2-\left(a_1+a_2\right)\left(b_1+b_2\right)c_2^2-b_3c_2^2-b_4c_2^2-2b_1c_3 \\
&\left.-2b_2c_3+\left(\left(a_1+2_2\right)\left(-3c_2^2+2c_3\right)\right)\right)+\sum_{m=3}^{6}\Phi_m\xi_k^m+O\left(\xi_k^7\right),
\end{aligned}
\tag{21}
$$

where $\Psi_i, \{i=1,2,3,4\}, \Phi_m, \{m=3,4,5\}$ are multivariate polynomial that depends on c_j for $\{2\leq j\leq 6\}$ and the parameters a_i, b_i for $\{1\leq i\leq 4\}$.
Substitute (6), (9), and (20) into the second step of (17), correspond to the error equation below:

$$
\begin{aligned}
x_{k+1} = {}&\delta-\left(\left(a_1+a_2-b_1-b_2\right)c_2^2\xi_k^3\right. \\
&-c_2\left(\left(1-a_3-a_4-5b_1-b_1^2-7b_2-2b_1b_2-b_2^2+a_1\left(5+b_1+b_2\right)\right.\right. \\
&+a_2\left(7+b_1+b_2\right)+b_3+b_4c_2^2 \quad + \quad 4\left(\left(-a_1-a_2+b_1+b_2\right)c_3\right)\right)\xi_k^4 \\
&+\sum_{i=5}^{6}\Gamma_i\xi_k^i+O\left(\xi_k^7\right);
\end{aligned}
\tag{22}
$$

where Γ_i are multi-variable polynomial expressed in c_j for $\{2 \leq j \leq 6\}$ and the parameters a_i, b_i for $\{1 \leq i \leq 4\}$. It is required that the coefficient of the errors ξ_k^i for $3 \leq i \leq 5$ vanish if the IM (21) is to converge to δ with order six. This is achievable when all the coefficient of ξ_k^i are set to zero. When solved in terms of a_1 and a_3, the following relations are obtain: $a_2 = -6 - a_1, a_4 = a_3 - 2a_1 - 1, b_1 = a_1 - 2, b_2 = -4 - a_1, b_3 = a_3 + 4, b_4 = -2 - a_3 - 2a_1$. When these relations are substituted in (21), the corresponding error equation is obtained as:

$$x_{k+1} = \delta + c_2^2 c_4 \xi_k^6 + O\left(\xi_k^6\right). \tag{23}$$

This completes the proof.

□

Remark 2.3. The substitution of the parameters (a_i and b_i) relations : $a_2 = -6-a_1, a_4 = -a_3-2a_1-1, b_1 = a_1-2, b_2 = -4-a_1$,$b_3 = 4+a_3$ and $b_4 = -2-a_3-2a_1$ into (17) results to the family of two parameter Iterative structure:

$$\begin{cases} x_{k+1} = x_k - \dfrac{\psi\left(x_k\right)}{\psi'\left(x_k\right)} - \dfrac{\psi'\left(y_k\right)}{\psi'\left(x_k\right)} H\left(s,u\right); \\ H\left(s,u\right) = \dfrac{1 + \left[a_1 - \left(6 + a_1\right)\dfrac{\psi'(y_k)}{\psi'(x_k)}\right]\dfrac{\psi(y_k)}{\psi(x_k)} + \left[a_3 - \left(a_3 + 2a_1 - 1\right)\dfrac{\psi'(y_k)}{\psi'(x_k)}\right]\left(\dfrac{\psi(y_k)}{\psi(x_k)}\right)^2}{1 + \left[\left(a_1 - 2\right) - \left(a_1 + 4\right)\dfrac{\psi'(y_k)}{\psi'(x_k)}\right]\dfrac{\psi(y_k)}{\psi(x_k)} + \left[\left(a_3 + 4\right) - \left(a_3 + 2a_1 + 2\right)\dfrac{\psi'(y_k)}{\psi'(x_k)}\right]\left(\dfrac{\psi(y_k)}{\psi(x_k)}\right)^2}. \end{cases} \tag{24}$$

The iterative structure (23) requires evaluation of four different functions in a complete cycle. Consequently, its EI is 1.5681.

Remark 2.4. For $a_1 = 1$ and $a_3 = -4$, a concrete member of (23) denoted M_2 is obtained as:

$$\begin{cases} x_{k+1} = x_k - \dfrac{\psi\left(x_k\right)}{\psi'\left(x_k\right)} - \dfrac{\psi'\left(y_k\right)}{\psi'\left(x_k\right)} H\left(s,u\right); \\ H\left(s,u\right) = \dfrac{1 + \left[1 - 7\frac{\psi'(y_k)}{\psi'(x_k)}\right]\frac{\psi(y_k)}{\psi(x_k)} - \left[4 - 3\frac{\psi'(y_k)}{\psi'(x_k)}\right]\left(\frac{\psi(y_k)}{\psi(x_k)}\right)^2}{1 - \left[1 + 5\frac{\psi'(y_k)}{\psi'(x_k)}\right]\frac{\psi(y_k)}{\psi(x_k)}}. \end{cases} \tag{25}$$

3. Numerical Implementation

This section presents the computational implementation of the developed methods on some real life problems expressed in NLM. To appreciate the developed methods effectiveness, their computational results are compared with the DNM (2) and method developed in Lee and Kim [6] put forward as:

$$\begin{cases} x_{k+1} = x_k - \dfrac{\psi(x_k)}{\psi'(x_k)} - \dfrac{\psi'(y_k)}{\psi'(x_k)} M(s,u); \\ M(s,u) = 1 + 2\left[1 - \dfrac{\psi'(y_k)}{\psi'(x_k)}\right]\dfrac{\psi(y_k)}{\psi(x_k)} - \left[1 + 2\dfrac{\psi'(y_k)}{\psi'(x_k)}\right]\left(\dfrac{\psi(y_k)}{\psi(x_k)}\right)^2. \end{cases} . \quad (26)$$

The MAPLE 2017 software environment was used to write and execute all computation programs for the developed methods and methods compared. The error bound and precision digits used are $\epsilon = 10^{-200}$ and 2000 significant figures respectively. For comparison, the number of iterations required by method to achieve convergence N, residual errors $|\psi(x_i)|$ and computational order of convergence ρ_{coc} in [11] given as:

$$\rho_{coc} = \frac{\log_{10} |\psi(x_{k+1})| / |\psi(x_k)|}{\log_{10} |\psi(x_k)| / |\psi(x_{k-1})|}. \quad (27)$$

were used. The following test problems $\psi_i(x) = 0$ also used in ([8], [9], [13]) are utilised for computational test.

Example 3.1. (Projectile motion [13])
$\psi_1(x) = x^3 - 9x + 1, \quad x_0 = 2.5, \quad \delta = 2.9428\ldots.$

Example 3.2. (Pollutant Concentration [13])
$\psi_2(x) = 2x - lnx - 7, \quad x_0 = 4.0\ , \quad \delta = 4.2199\ldots.$

Example 3.3. (Anti-symmetric buckling [13])
$\psi_3(x) = e^x + x - 20\ , \quad x_0 = 2.0\ , \quad \delta = 2.842\ldots.$

Example 3.4. (Mass of a Jumper [13])
$\psi_4(x) = sinx - x + 1, \quad x_0 = -1.0 \quad \delta = 1.9345\ldots.$

Example 3.5. (Colebrook-White equation [8])
$\psi_5(x) = \sqrt{\frac{1}{f}} + 2log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{R\sqrt{f}}\right)$, using $\epsilon/D = 10^{-4}$, $R = 10^5$,
$x_0 = 0.002 \quad \delta = 0.0041\ldots$

Example 3.6. (Population growth [9])
$\psi_6(x) = 1586000 - \frac{435000}{x}(e^x - 1) - 1000000e^x, \quad x_0 = 0.5 \quad \delta = 0.1173\ldots$

Example 3.7. (Van der Waals equation [13])
$\psi_7(x) = 0.986x^3 - 5.181x^2 + 9.067x - 5.289, \quad x_0 = 2.0 \quad \delta = 1.9298\ldots$

Example 3.8. (Reactor concentration [9]
$\psi_9(x) = -0.75e^{-0.05x} + 1, \qquad x_0 = 1.0 \qquad \delta = -5.753\ldots$

The computational results of the developed and compared methods on the tested problems are presented in Table 1. Observe that the developed methods solved all the test problems with computational CO that agrees with the theoretical CO established in section 2. This is evidence in the last column of Table 1. In addition, the residual errors $|\psi(x_i)|$ obtained from the test problems using M_1 and M_2 are in most cases smaller than that of the compared methods.

4. Conclusion

This manuscript put forward two families of IM for estimating the solution of nonlinear models. The methods are modification of the DNM and designed by the introduction of weight functions $G(s,u)$ and $H(s,u)$ that are quotients of two second order bi-variate power series. The theoretical and computational analysis done on both methods confirmed that they are of CO six requiring same number of function evaluation as the DNM. Further, the computational test and comparison shows that the methods developed here in are effective for solving NLM.

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Table 1. Methods results comparison for models $\psi_1 - \psi_8$.

$Methods$	N	$\|\psi(x_1)\|$	$\|\psi(x_2)\|$	$\|\psi(x_3)\|$	$\|\psi(x_4)\|$	$\|\psi(x_5)\|$	ρ_{coc}
DNM	5	$2.0e-1$	$4.5e-8$	$1.1e-34$	$4.7e-141$	$1.4e-566$	4.0
LSM	4	$8.0e-1$	$6.1e-8$	$1.7e-50$	$8.4e-306$	-	6.1
M_1	4	$2.7e-1$	$1.1e-11$	$5.9e-74$	$1.5e-447$	-	6.0
M_2	4	$9.2e-3$	$1.1e-23$	$3.3e-170$	$8.6e-1196$	-	7.0
DNM	4	$1.9e-8$	$1.1e-37$	$9.4e-155$	$5.7e-623$	-	4.0
LSM	3	$5.8e-11$	$4.8e-70$	$1.5e-424$	-	-	6.1
M_1	3	$4.5e-11$	$8.6e-71$	$4.4e-429$	-	-	6.0
M_2	3	$2.8e-11$	$3.4e-72$	$1.1e-437$	-	-	6.1
DNM	6	1.4	$5.8e-5$	$2.0e-22$	$2.9e-92$	$1.2e-371$	4.0
LSM	5	19.82	$1.2e-1$	$3.5e-13$	$1.7e-82$	$2.4e-498$	6.1
M_1	4	1.32	$1.7e-7$	$1.3e-48$	$2.2e-295$	-	6.2
M_2	4	$5.3e-2$	$9.7e-17$	$3.8e-105$	$1.3e-635$	-	6.1
DNM	5	$1.3e-1$	$3.0e-6$	$1.4e-24$	$5.8e-98$	$1.9e-391$	4.0
LSM	5	3.4	$3.1e-4$	$1.5e-23$	$2.3e-134$	$2.4e-834$	6.0
M_1	5	$7.8e-1$	$7.6e-5$	$3.1e-28$	$1.5e-168$	$1.8e-1010$	6.0
M_2	4	$3.7e-1$	$6.8e-8$	$7.4e-47$	$1.2e-280$	-	6.0
DNM	5	$4.3e-1$	$1.8e-5$	$6.4e-23$	$9.4e-93$	$4.4e-372$	4.0
LSM	4	$1.8e-1$	$5.7e-10$	$6.9e-61$	$2.1e-366$	-	6.0
M_1	5	1.42	$5.1e-4$	$3.6e-25$	$4.8e-152$	$2.7e-913$	6.0
M_2	4	$1.2e-1$	$2.0e-11$	$3.9e-70$	$1.9e-422$	-	6.0
DNM	5	2408.6	$1.4e-6$	$1.6e-43$	$2.7e-191$	$2.3e-782$	4.1
LSM	4	341.8	$6.6e-17$	$3.3e-129$	$5.1e-803$	-	6.2
M_1	4	183.3	$7.2e-19$	$2.5e-141$	$5.0e-876$	-	6.2
M_2	4	26.0	$5.6e-25$	$5.4e-179$	$4.6e-1103$	-	6.2
DNM	5	$1.6e-4$	$2.2e-10$	$8.4e-34$	$1.7e-127$	$3.3e-502$	4.0
LSM	4	$3.2e-6$	$3.8e-24$	$9.2e-132$	$1.9e-777$	-	5.9
M_1	4	$6.3e-5$	$1.2e-15$	$7.7e-80$	$4.2e-465$	-	5.8
M_2	4	$9.6e-6$	$1.5e-24$	$4.1e-156$	$4.0e-1077$	-	6.9
DNM	5	$2.0e-3$	$2.0e-12$	$2.1e-48$	$2.4e-192$	$3.8e-768$	4.0
LSM	4	$1.0e-3$	$3.0e-19$	$1.9e-112$	$1.2e-671$	-	6.0
M_1	4	$6.5e-4$	$8.5e-21$	$4.5e-122$	$9.0e-730$	-	6.0
M_2	4	$1.7e-5$	$2.7e-31$	$4.0e-186$	$4.0e-1115$	-	6.0