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Exact controllability and continuous dependence of solution of a conformable fractional control system

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Abstract. The exact controllability of a conformable fractional differential system is established in this paper. The system is described by a non-densely defined linear part satisfying the Hille Yosida condition, and a control term appearing in the nonlinear part. The existence of mild solution and exact controllability is proved by Banach fixed point theorem for the system with non-local conditions and deviated argument. The continuous dependence of the mild solution is also studied. An example is discussed to illustrate the results.

1. Introduction

In this paper existence, continuous dependence of mild solution and exact controllability of a class of conformable fractional control system is discussed. The linear part of the system is non-densely defined with the control parameter also appearing in the nonlinear part. Deviated argument and non-local conditions are used to capture physical conditions. Here the following conformable fractional differential control system is studied in a Hilbert space $(X, ||.||)$.

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + Bu(t) + f(t, x(c(x(t), t)), u(t)), \ 0 < t \leq a, \\ x(0) &= x_0 + g(x) \end{aligned} \tag{1}$$

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where $0 < \alpha < 1$, D^α denotes the conformable time fractional derivative, $A : D(A) \subset X \to X$ is a non-densely defined linear operator i.e. $\overline{D(A)} \neq X$ and satisfying Hille - Yosida condition. B is a bounded linear operator from the Hilbert space U to X with the control function $u \in L^2([0,a],U)$.

Non-local initial conditions model various physical situations better than usual initial conditions. For more applications of non-local conditions one may refer [5]. Fractional derivatives are used in place of the classical derivatives to incorporate memory properties inherent in many systems. For more details see [1, 2, 3, 4, 6] .

Oflate Khalil et al.[9] introduced the concept of a new fractional derivative called conformable fractional derivative. The new definition is compatible with classical derivative. Its many applications arise in mechanics, electronics, anomalous diffusion etc. The novel definition extends the usual limit definition of classical derivative . For more details see[5, 9].

Fractional derivatives defined as integrals as in Caputo or Riemann derivatives, contradict the notion of locality of classical derivatives. As early as 17th century, the concept of derivative is local, contrary to globality of fractional derivatives which are defined in terms of integrals. Fractional derivatives in terms of integrals have therefore being claimed as not derivatives in the strict sense. Derivatives represent instances, particular magnitudes rather than intervals. Fractional models are one particular set of models illustrating fractional behaviours.

Generally A is densely defined operator on a associated Banach space. However, various real world situations are better depicted by non-dense operators. As an example one can see that in a heat equation with Dirichlet condition on $[0,1]$ if $A = \frac{\partial^2}{\partial^2 x}$ on $C([0,1];R)$ with supremum norm over the domain

$$D(A) = \{u \in C^2([0,1];R) : u(0) = u(1) = 0\}$$

is not dense in $C([0,1];R)$. The main objective of this paper is to fill these gap in the study of controllability of conformable fractional control systems with nondensely defined operator, deviated argument and nonlocal condition.

2. Preliminaries

Conformable fractional derivative of order α of a function y at $t > 0$ is defined by

$$\frac{d^\alpha y(t)}{dt^\alpha} = \lim_{\epsilon \to 0} \frac{y(t + \epsilon t^{1-\alpha}) - y(t)}{\epsilon}.$$

Conformable fractional integral of order α of a function y is defined as

$$I^\alpha y(t) = \int_0^t s^{\alpha - 1} y(s) ds.$$

If y is a continuous function in the domain of I^α, we have

$$\frac{d^\alpha(I^\alpha y(t))}{dt^\alpha} = y(t).$$

If y is differentiable, we have

$$I^\alpha \frac{d^\alpha y(t)}{dt^\alpha}(t) = y(t) - y(0)$$

Hypothesis $(H0)$: [9] The non-dense operator $A : D(A) \subset X \to X$ satisfies Hille-Yosida condition i.e. $\exists\ \overline{M} \geq 0,\ w \in \mathbb{R},\ (w, \infty) \subset \rho(A)$ such that

$$\sup\{(\lambda - w)^n \|R(\lambda, A)^n\|, n \in \mathbb{N},\ \lambda > w\} \leq \overline{M},$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$.
When the Hille-Yosida condition is satisfied the non-dense operator A generates a non-degenerate, locally Lipschitz continuous integrated semigroup
Let us define A_0 on $D(A_0) \subset \overline{D(A)}$ as

$$A_0 x = Ax,$$

and

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}.$$

Then A_0 generates a family $\{T(t)\}_{t\geq 0}$ of C_0 semigroup on $\overline{D(A)}$.
Let

$$C([0, a]; \overline{D(A)}) = \{x : [0, a] \to \overline{D(A)} : x \ is \ continuous \ on \ [0, a]\},$$

$$C_L([0, a]; \overline{D(A)}) = \{x : [0, a] \to C([0, a], \overline{D(A)}) : \|x(t) - x(s)\| \leq L|t - s|\}.$$

Clearly $C_L([0, a]; \overline{D(A)})$ is a Banach space,where $\|x\|_a = \sup_{t\in[0,a]} \|x(t)\|$. Also let U_c denote the space of all continuous functions from $[0, a]$ to U. Then the product space $C_L([0, a]; \overline{D(A)}) \times U_c$ is a Banach space with norm defined as

$$\|(.,.)\|_{C_L([0,a];\overline{D(A)})\times U_c} = \|.\|_{C_L([0,a];\overline{D(A)})} + \|.\|_{U_c}$$

Let us define the controllability operator

$$G_r^t = \int_r^t s^{q-1}[T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B][T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B]^* ds,$$

$0 \leq r \leq t \leq a$. Now consider the following hypotheses:

(H1) The function $f : [0, a] \times C_L([0, a]; X) \times U_{ad} \to C_L([0, a]; X)$ is a continuous function such that $\forall t \in [0, a],\ x, y \in C_L([0, a]; \overline{D(A)}), u, w \in U_{ad}$

$$\|f(t, x, u) - f(t, y, w)\|_a \leq L_f[\|x - y\|_a + \|u - w\|_a],$$

where $L_f \geq 0$

(H2) $\exists\ L_b > 0$ such that $\forall t \in [0, a]\ \|c(x(t), t) - c(y(t), t)\| \leq L_c\|x(t) - y(t)\|$.

(H3) $\exists\ L_T > 0$ such that $\|T(t)\| \leq L_T\ \forall\ t \in [0, a]$.

(H4) $\exists\ L_g > 0$ such that $\forall t \in [0, a]\ \|g(x(t)) - g(y(t))\| \leq L_g\|x(t) - y(t)\|$.

(H5) (G_0^a) is coercive, which implies that $\exists\ \mu > 0$ such that $\langle G_0^a x, x\rangle \geq \mu\|x\|^2$, hence $(G_0^a)^{-1}$ exists and $\|(G_0^a)^{-1}\| \leq \frac{1}{\mu}$

Definition 1. *The system (1) is said to be controllable on the interval* $[0,a]$ *if for every initial function* $x(0) \in \overline{D(A)}$ *and* $x_a \in \overline{D(A)}$ $\exists$ *a control* $u \in L_2([0,T];U)$ *such that the integral solution* $x(.)$ *of (1) satisfies* $x(a) = x_a$.

3. Main Result

In this section existence and uniqueness of the mild solution and exact controllability of the control system is established. Then the continuous dependence of the integral solution is also studied.

Definition 2. *A function* $x(t) \in C_L([0,a];\overline{D(A)})$ *is called a mild solution of (1) if it satisfies the integral equation*

$$\begin{aligned} x(t) &= T(\frac{t^\alpha}{\alpha})(x_0 + g(x)) + \int_0^t s^{\alpha-1} T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}) \\ &\times\ (f(s, x(c(x(s),s)), u(s)) + Bu(s))ds,\ \forall t \in [0,a]. \end{aligned} \tag{2}$$

Lemma 3.1. [9] *Assuming that* X *and* U *are Hilbert spaces and* A_0 *is an infinitesimal generator of* C_0 *semigroup and* B *is a bounded linear operator from* X *to* U *then we have*

$$\|G_0^t\| \leq \|G_0^a\|,\ 0 \leq t \leq a$$

Proof. It is clear that $G_0^t = (G_0^t)^*$ and $\forall\ x \in X,\ \langle G_0^t x, x\rangle \geq 0$. Also $\|G_0^t\| = \sup_{\|x\|\leq 1} |\langle G_0^t x, x\rangle|$. So,

$$\begin{aligned} \langle G_0^a x, x\rangle &= \langle \int_0^a s^{\alpha-1}[T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B][T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B]^* dsx, x\rangle \\ &= \langle \int_0^t s^{\alpha-1}[T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B][T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B]^* dsx, x\rangle \\ &+ \langle \int_t^a s^{\alpha-1}[T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B][T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B]^* dsx, x\rangle \\ &= \langle G_0^t x, x\rangle + \langle \int_t^a s^{\alpha-1}[T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B][T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B]^* dsx, x\rangle \end{aligned} \tag{3}$$

Now,

$$\begin{aligned} &\langle \int_t^a s^{q-1}[T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B][T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B]^* dsx, x\rangle \\ &= \int_t^a s^{\alpha-1}\langle [T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B]^* x, [T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B]^* x\rangle ds \\ &= \int_t^a s^{\alpha-1}\|[T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})B]^* x\| ds \geq 0 \end{aligned} \tag{4}$$

Therefore $\langle G_0^t x, x\rangle \leq \langle G_0^a x, x\rangle$, implies $\|G_0^t\| \leq \|G_0^a\|,\ 0 \leq t \leq a$ □

Let $p \in \overline{D(A)}$ then under hypothesis $(H5)$ define the control function as

$$\begin{aligned} U(t) &= B^*T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[\Omega_0^a]^{-1}(p - T(\frac{a^\alpha}{\alpha})(x_0 + g(x)) \\ &- \int_0^a s^{\alpha-1}T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})f(s, x(c(x(s), s)), u(s))ds). \end{aligned} \quad (5)$$

Let

$$\begin{aligned} X(t) &= T(\frac{t^\alpha}{\alpha})(x_0 + g(x)) + G_0^t T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[G_0^T]^{-1}(p - T(\frac{a^\alpha}{\alpha})(x_0 + g(x))) \\ &- G_0^t T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[G_0^T]^{-1} \\ &\times \int_0^a s^{\alpha-1}T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})f(s, x(c(x(s), s)), u(s))ds \\ &+ \int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})f(s, x(c(x(s), s)), u(s))ds. \end{aligned} \quad (6)$$

Lemma 3.2. *By hypotheses* $(H1)-(H5)$, *the operator* $K : C_L(0, a; \overline{D(A)}) \times U_c \to C_L(0, a; \overline{D(A)}) \times U_c$ *defined as* $K(x, u)(t) = (X(t), U(t)),\ 0 \leq t \leq a$ *satisfies the estimate*

$$\|K(y, w) - K(x, u)\| \leq \frac{a^\alpha}{\alpha}[(1 + \frac{\|G_0^a\|}{\mu}L_T)L_T L_f + 1](\|y - x\| + \|u - w\|)$$

whenever $LL_b \leq 1,\ L_T L_g \leq L_f \frac{a^\alpha}{\alpha}$ *and* $L_T L_g + L_g L_T^2 \frac{G_0^T}{\mu} \leq \frac{a^\alpha}{\alpha}$.

Proof. Let $(y, w),\ (x, u) \in\ C_L([0, a]; \overline{D(A)}) \times U_c$, with $K(y, w) = (Y, W)$, $K(x, u) = (X, U)$. Then

$$\|K(y, w) - K(x, u)\| = \|Y - X\|_{\ C_L([0,a];\overline{D(A)})} + \|W - U\|_{U_c}.$$

Now let us estimate

$$\begin{aligned}
\|Y-X\| &= \sup_{t\in[0,a]}\|G_0^tT^*(\frac{a^\alpha}{\alpha}-\frac{t^\alpha}{\alpha})[\Omega_0^a]^{-1}[\times\int_0^a s^{\alpha-1}T(\frac{a^\alpha}{\alpha}-\frac{t^\alpha}{\alpha})\\
&\times\ (f(s,y(b(y(s),s)),w(s))-f(s,x(c(x(s),s)),u(s)))ds]\\
&+\ \int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})(f(s,y(c(y(s),s)),w(s))\\
&-\ f(s,x(c(x(s),s)),u(s)))ds+T(\frac{t^\alpha}{\alpha})(g(y)-g(x))\\
&+\ G_0^tT^*(\frac{a^\alpha}{\alpha}-\frac{t^\alpha}{\alpha})[G_0^a]^{-1}(T(\frac{t^\alpha}{\alpha})(g(x)-g(y)))\|\\
&\le \sup_{t\in[0,a]}(L_T+L_T^2\|G_0^t\|\|G_0^a\|^{-1})\\
&\times\ [\int_0^a s^{\alpha-1}\|(f(s,y(c(y(s),s)),w(s))\\
&-\ f(s,x(c(x(s),s)),u(s)))\|ds]\\
&+\ L_TL_g(\|y(s)-x(s)\|)+\frac{\|G^a\|}{\mu}L_T^2L_g(\|x(s)-y(s)\|)\\
&\le (L_T+L_T^2\frac{\|G_0^a\|}{\mu})L_f\int_0^a s^{\alpha-1}(LL_c\|x(s)-y(s)\|+\|w-u\|)ds\\
&+\ L_TL_g(\|y-z\|)+\frac{\|G_0^a\|}{\mu}L_T^2L_g\|x-y\|
\end{aligned}$$

$$\begin{aligned}
&\le [(1+\frac{\|G_0^a\|}{\mu}L_T)L_TL_f\frac{a^\alpha}{\alpha}(LL_c)+L_TL_g+L_T^2L_g\frac{\|G_0^a\|}{\mu}]\|x-y\|\\
&+\ [(1+\frac{\|G_0^a\|}{\mu}L_T)L_TL_f\frac{a^\alpha}{\alpha}]\|w-u\|\\
&= [\frac{a^\alpha}{\alpha}(1+\frac{\|G_0^a\|}{\mu}L_T)L_TL_f(LL_c)+L_TL_g+L_gL_T^2\frac{\|G_0^a\|}{\mu}]\|x-y\|\\
&+\ [\frac{a^\alpha}{\alpha}(1+\frac{\|G_0^a\|}{\mu}L_T)L_TL_f\|w-u\| \qquad (7)
\end{aligned}$$

since $LL_c<1$ and $L_TL_g+L_gL_T^2\frac{\|G_0^a\|}{\mu}\le\frac{a^\alpha}{\alpha}$

$$\begin{aligned}
&\le [\frac{a^\alpha}{\alpha}(1+\frac{\|G_0^a\|}{\mu}L_T)L_TL_f+\frac{a^\alpha}{\alpha}]\|y-x\|\\
&+\ [\frac{a^\alpha}{\alpha}(1+\frac{\|G_0^a\|}{\mu}L_T)L_TL_f]\|u-w\|+\frac{a^\alpha}{\alpha}]\|u-w\|\\
&= \frac{a^\alpha}{\alpha}[(1+\frac{\|G_0^a\|}{\mu}L_T)L_TL_f+1](\|x-y\|+\|u-w\|)\\
&= \frac{a^\alpha}{\alpha}M_0(\|x-y\|+\|u-w\|) \qquad (8)
\end{aligned}$$

where $M_0 = [(1 + \frac{\|G_0^\alpha\|}{\mu} L_T) L_T L_f + 1]$. Similarly,

$$\begin{aligned} \|U - W\| &= \sup_{t\in[0,a]} \|B^* T^* (\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[G_0^a]^{-1}[\{\int_0^a s^{\alpha-1}(f(s, y(c(y(s), s)), w(s)) \\ &- f((s, x(c(x(s), s)), u(s)))ds\} + T^*(\frac{a^\alpha}{\alpha})(g(y) - g(x))]\| \\ &\le \frac{L_c L_T}{\mu}[\int_0^a s^{\alpha-1}\|(f(s, y(c(y(s), s)), w(s)) \\ &- f((s, x(c(x(s), s)), u(s)))\|ds + L_T L_g\|y - x\|] \\ &\le \frac{L_c L_T}{\mu}[\frac{a^\alpha}{\alpha} L_f(LL_c\|y - x\| + \|w - u\|) + L_T L_g\|y - x\|] \\ &\le \frac{L_c L_T}{\mu}[\frac{a^\alpha}{\alpha} L_f(\|y - x\| + \|w - u\|) + \frac{a^\alpha}{\alpha} L_f\|y - x\|] \\ &\le \frac{L_c L_T}{\mu}[\frac{a^\alpha}{\alpha} L_f(\|y - x\| + \|w - u\|) \\ &+ \frac{a^\alpha}{\alpha} L_f(\|y - x\| + \|u - w\|)] \\ &\le \frac{L_c L_T}{\mu}[2\frac{a^\alpha}{\alpha} L_f(\|y - x\| + \|w - u\|)] \end{aligned} \tag{9}$$

Therefore from (8) and (9) we get

$$\|K(x, u) - K(y, w)\| \le \frac{a^\alpha}{\alpha}[(M_0 + \frac{L_c L_T}{\mu}(2L_f)](\|x - y\| + \|u - w\|) \tag{10}$$

□

Lemma 3.3. *Assuming that lemma(3.2) holds, the operator K defined as $K(x, u)(t) = (X(t), U(t))$, $0 \le t \le a$ maps $C_L(0, a; \overline{D(A)}) \times U_c$ into $C_L(0, a; \overline{D(A)}) \times U_c$ and has a unique fixed point $(x, u) \in C_L(0, a; \overline{D(A)}) \times U_c$ if $\frac{a^\alpha}{\alpha}[M_0 + \frac{L_b L_T}{\mu}(2L_f)] < 1$*

Proof. Since the product space $C_L([0, a]; \overline{D(A)}) \times U_c$ is a Banach space endowed with the norm

$$\|(.,.)\|_{C_L([0,a];\overline{D(A)})\times U_c} = \|.\|_{C_L([0,a];\overline{D(A)})} + \|.\|_{U_c}$$

and from lemma (3.2) we get

$$\|K(x, u) - K(y, w)\| \le \frac{a^\alpha}{\alpha}[M_0 + \frac{L_b L_T}{\mu}(2L_f)](\|x - y\| + \|u - w\|),$$

so clearly the operator K is a contraction mapping if $\frac{a^\alpha}{\alpha}[M_0 + \frac{L_b L_T}{\mu}(2L_f)] < 1$. Therefore it has a unique fixed point $(x, u) \in C_L(0, a; \overline{D(A)}) \times U_c$. □

Theorem 3.4. *Assuming that lemma(3.3) holds, the non-local conformable fractional differential system (1) is exactly controllable on $[0, a]$.*

PROOF. To prove the system (1) is exactly controllable on $[0, a]$, we need to show that for any $p \in D(A)$, $\exists u \in U_c$ so that $p = x(T)$. From (5), we get

$$
\begin{aligned}
u(t) &= B^*T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[\Omega_0^a]^{-1}[p - T(\frac{a^\alpha}{\alpha})(x_0 + g(x)) \\
&- \int_0^a s^{\alpha-1}T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})f(s, x(c(x(s), s)), u(s))ds].
\end{aligned} \tag{11}
$$

let us substitute (11) in (2), so that,

$$
\begin{aligned}
x(t) &= T(\frac{t^\alpha}{\alpha})(x_0 + g(x)) + \int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})BB^*T^*(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}) \\
&\times T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[\Omega_0^a]^{-1}\{p - T(\frac{a^\alpha}{\alpha})(x_0 + g(x(r)))\}ds \\
&- \int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})BB^*T^*(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[\Omega_0^a]^{-1} \\
&\times (\int_0^a r^{\alpha-1}T(\frac{a^\alpha}{\alpha} - \frac{r^\alpha}{\alpha})f(r, x(c(x(r), r)), u(r))dr)ds \\
&+ \int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})f(s, x(c(x(s), s)), u(s))ds. \\
&= T(\frac{t^\alpha}{\alpha})(x_0 + g(x)) + \int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})f(s, x(c(x(s), s)), u(s))ds \\
&+ \Omega_0^t T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[\Omega_0^a]^{-1}\{p - T(\frac{a^\alpha}{\alpha})(x_0 + g(x(r)))\} \\
&- \int_0^a [\int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})BB^*T^*(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[\Omega_0^a]^{-1}ds] \\
&\times T(\frac{a^\alpha}{\alpha} - \frac{r^\alpha}{\alpha})f(r, x(c(x(r), r)), u(r))r^{\alpha-1}dr \\
&= T(\frac{t^\alpha}{\alpha})(x_0 + g(x)) + \int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})f(s, x(c(x(s), s)), u(s))ds \\
&+ \Omega_0^t T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[\Omega_0^a]^{-1}\{p - T(\frac{a^\alpha}{\alpha})(x_0 + g(x(r)))\} \\
&- \int_0^a \Omega_0^t T^*(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})[\Omega_0^a]^{-1}r^{\alpha-1}T(\frac{a^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})f(r, x(c(x(r), r)), u(r))dr
\end{aligned} \tag{12}
$$

Now let us take $t = a$, then

$$\begin{aligned} x(a) &= T(\frac{a^\alpha}{\alpha})(x_0 + g(x)) + \int_0^a s^{\alpha-1}T(\frac{a^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})f(s, x(c(x(s), s)), u(s))ds \\ &+ p - T(\frac{a^\alpha}{\alpha})(x_0 + g(x)) \\ &- \int_0^a r^{\alpha-1}T(\frac{a^\alpha}{\alpha} - \frac{r^\alpha}{\alpha})f(r, x(c(x(r), r)), u(r))dr \\ &= p. \end{aligned} \tag{13}$$

This implies that $\forall p \in \overline{D(A)}$, $\exists u \in U_c$ such that it transfers $x(0)$ to p in time a such that $p = x(a)$. Hence the system (1) is exactly controllable. □

3.1. Continuous Dependence. Now continuous dependence is studied to understand the effect of the non-local condition on the stability of the mild solution.

Theorem 3.5. *Let $x,\ y$ be the solutions with corresponding initial nonlocal conditions associated to $x_0,\ x_1$ respectively. Let $u_0,\ u_1 \in U_c$. Assuming the hypotheses $(H0 - H5)$ hold, and if $(L_T L_g + L_T L_f \frac{a^\alpha}{\alpha}(LL_b)) < 1$ with*

$$\begin{aligned} x(t) &= T(\frac{t^\alpha}{\alpha})(x_0 + g(x)) + \int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}) \\ &\times (f(s, x(c(x(s), s)), u_0(s)) + Bu_0(s))ds,\ \forall t \in [0, a]. \end{aligned} \tag{14}$$

and

$$\begin{aligned} y(t) &= T(\frac{t^\alpha}{\alpha})(x_1 + g(y)) + \int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}) \\ &\times (f(s, y(c(y(s), s)), u_1(s)) + Bu_1(s))ds,\ \forall t \in [0, a]. \end{aligned} \tag{15}$$

Then $\exists$ constants $C_0,\ C_1 > 0$ such that $\|x - y\|_{C_L([0,a];\overline{D(A)})} \leq C_0|x_0 - x_1| + C_1\|u_0 - u_1\|_{U_c}$

Proof.

$$\begin{aligned} \|x - y\| &\leq \|T(\frac{t^\alpha}{\alpha})\||x_0 - x_1| + \|T(\frac{t^\alpha}{\alpha})\|\|g(x) - g(y)\| \\ &+ \|\int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})(Bu_0(s) - Bu_1(s))ds\| \\ &+ \|\int_0^t s^{\alpha-1}T(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})(f((s, x(c(x(s), s)), u_0(s)) \\ &- f((s, y(c(y(s), s)), u_1(s)))ds\| \\ &\leq L_T|x_0 - x_1| + L_T L_g\|x - y\| + L_T L_f\frac{a^\alpha}{\alpha}(LL_c)\|x - y\| \\ &+ (L_T L_f\frac{a^\alpha}{\alpha})\|u_0 - u_1\| + \|B\|\frac{a^\alpha}{\alpha}\|u_0 - u_1\| \end{aligned} \tag{16}$$

This implies

$$\begin{aligned} \|x-y\| &\le \frac{L_T}{1-L_TL_g-L_TL_f\frac{a^\alpha}{\alpha}(LL_c)}|x_0-x_1| \\ &+ \frac{[(L_TL_f+\|B\|)\frac{a^\alpha}{\alpha}]}{1-L_TL_g-L_TL_f\frac{a^\alpha}{\alpha}(LL_c)}\|u_0-u_1\| \\ &= C_0\|x_0-x_1|+C_1\|u_0-u_1\| \end{aligned} \tag{17}$$

where $C_0 = \frac{L_T}{1-L_TL_g-L_TL_f\frac{a^\alpha}{\alpha}(LL_c)}$ and $C_1 = \frac{[(L_TL_f+\|B\|)\frac{a^\alpha}{\alpha})]}{1-L_TL_g-L_TL_f\frac{a^\alpha}{\alpha}(LL_b)}$. Hence proved. □

4. Example

Example 4.1. As an example of a conformable fractional control system one may consider the system

$$\begin{aligned} D^\alpha x(y,t) &= Ax(y,t)+u(t) \\ &+ \int_0^x K(s,x)(x(y,\|x(y,t)\|)+u(t))ds, \\ & \quad 0<t\le a,\ y\in(0,\pi) \\ x(0,t) &= x(\pi,t)=0 \\ x(y,0) &= x_0(y)+g(x(y,t)),\ y\in(0,\pi) \end{aligned} \tag{18}$$

With operator A satisfying Hille-Yosida condition and the hypotheses (H1)- (H2) on being satisfied, it can be shown that results in Theorem (3.4)holds.

Example 4.2. This example is studied as a particular case of the class of examples as in (4.1).

$$\begin{aligned} D^{1/2}x(y,t) &= \frac{\partial^2}{\partial y^2}x(y,t)+u(t) \\ &+ \int_0^x K(s,x)(x(y,\|x(y,t)\|)+u(t))ds, \\ & \quad 0<t\le a,\ y\in(0,\pi) \\ x(0,t) &= x(\pi,t)=0 \\ x(y,0) &= x_0(y)+g(x(y,t)),\ y\in(0,\pi) \end{aligned} \tag{19}$$

Let $X=C([0,\pi],\mathbb{R})$ and $Ax := \frac{\partial^2}{\partial y^2}$ on the domain

$$D(A) = \{x(.)\in X : \ \frac{\partial^2}{\partial y^2} \ \in X,\ x(0)=x(\pi)=0\}.$$

Clearly $\overline{D(A)} \neq X$. Hence A is nondensely defined on X. Here $\rho(A) \supseteq (0,\infty)$, $\|\lambda I-A)^{-1}\| \le \frac{1}{\lambda},\ \lambda>0$, so A generates C_0 semigroup $\{T(t)\}$ on $\overline{D(A)}$. Let

$$C_L([0,a],\overline{D(A)}) = \{x : [0,a]\to C([0,a],\overline{D(A)}) : \|x(t)-x(s)\| \le L|t-s|\}.$$

Set $x(t) := x(y,t)$. Define $f : [0,a] \times C_L([0,a], \overline{D(A)}) \times U \to X$ as

$$f(t, x(b(x(t),t)), u(t)) = \int_0^x K(s,x)(x(y, \|x(y,t)\|) + u(t))ds,$$

as

$$x(t) := x(y,t)$$

So

$$x(y, \|x(y,t)\|) = x(\|x(y,t)\|) = x(b(x(t),t)))$$

where $b(x(t),t) = \|x(y,t)\|$ and $Bu(t) = u(t)$. Hypotheses $(H1)$ and $(H2)$ are verified as below

$$\begin{aligned} \|f(t, x(b(x(t),t)), u(t)) &- f(t, z(b(z(t),t)), v(t))\| \\ &\leq \int_0^x K(s,x)\|x(\|x(y,t)\|) - z(\|z(y,t)\|)\|ds \\ &+ \int_0^x K(s,x)(\|u(t) - v(t)\|)ds \\ &\leq \int_0^x K(s,x)(\|x(t)\| - \|z(t)\|)ds \\ &+ \int_0^x K(s,x)\|u(t) - v(t)\|ds \\ &\leq \int_0^x K(s,x)(\|x(t) - z(t)\|)ds \\ &+ \int_0^x K(s,x)\|u(t) - v(t)\|ds \end{aligned}$$

So the hypotheses $(H1)$ and $(H2)$ are satisfied. Hence the (19) can be reduced to the form

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + Bu(t) + f(t, x(b(x(t),t)), u(t)),\ 0 < t \leq a, \\ x(0) &= x_0 + g(x) \end{aligned} \tag{20}$$

By Theorem(3.4) the system (19) is therefore exactly controllable on $[0,a]$.

5. Conclusion

Exact controllability and continuous dependence mild solution of a class of conformable fractional control system of order $\alpha \in (0,1]$ is established. Generally the authors use a densely defined operator A. But in this paper A is belongs to broader class of operators called non-dense operators. Moreover non-local initial conditions and deviated argument is used to incorporate the properties of the control system that are not captured by usual initial conditions. The control operator is included in the nonlinear term as well to study nonlinear control systems. Eventually, an example is given to illustrate the main result.

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