Mathematical Analysis and its Contemporary Applications
Volume 4, Issue 2, 2022, 11–16
doi: 10.30495/maca.2022.1946498.1040
ISSN 2716-9898

Automatic continuity of almost Jordan derivations on special Jordan Banach algebras

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Abstract. The following is the question form of Kaplansky conjecture of 1958. Is every derivation on semisimple Banach algebra continuous? Kaplansky conjecture was proved by Johnson and Sinclair in 1968. The concept of almost Jordan derivations on Jordan Banach algebras is introduced in this article. Also, Kaplansky conjecture is extended to Jordan Banach algebras as an open question: Is every almost Jordan derivations on semisimple Jordan Banach algebras continuous?. Moreover, a partial answer to this open question is derived in the sense that every almost Jordan derivation T on semisimple special Jordan Banach algebras Ω^+, with an additional condition on Ω^+, is continuous.

1. Introduction

We provide a brief outline of definitions and known outcomes in this section. For more details, one may refer to [2]. All vector spaces are considered over the complex field, and we assume that all algebras are unital. All algebras which are to be considered are associative algebras except Jordan algebras and Jordan Banach algebras.

Definition 1.1. [2] A Banach algebra Ω is a complete normed algebra, where a normed algebra Ω is an algebra with a norm $||.||$, which also satisfies $||p.q|| \leq ||p||.||q||$, $\forall\ p, q \in \Omega$.

Definition 1.2. An algebra with a Hausdorff topology is called a topological algebra, if all algebraic operations are jointly continuous.

2010 *Mathematics Subject Classification.* Primary: 46H40; Secondary: 46H05.
Key words and phrases. Automatic continuity, Almost derivation, Banach algebras.
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Definition 1.3. [2] The Jacobson radical $rad(\Omega)$ of an algebra Ω is the intersection of all maximal right(or left) ideals. An algebra is said to be semisimple, if $rad(\Omega) = \{0\}$.

Definition 1.4. [2] The spectrum $\sigma_\Omega(p)$ of an element p of an algebra Ω is the set of all complex numbers λ such that $\lambda.1 - p$ is not invertible in Ω. The spectral radius $r_\Omega(p)$ of an element $p \in \Omega$ is defined by $r_\Omega(p) = \sup\{|\lambda| : \lambda \in \sigma_\Omega(p)\}$.

If $(\Omega, ||.||)$ is a Banach algebra, then $r_\Omega(p) = \lim_{n\to\infty} ||p^n||^{\frac{1}{n}}$. Also, for any algebra Ω, we have $rad(\Omega) = \{p \in \Omega : r_\Omega(pq) = 0, \text{ for every } q \in \Omega\}$. See [2].

Definition 1.5. [2] If $T : \Omega \to \Gamma$ is a linear map from a Banach algebra Ω to a Banach algebra Γ, then the separating space of T is defined as the set
$S(T) = \{q \in \Gamma : \text{ there exists } (p_n)_{n=1}^{\infty} \text{ in } \Omega \text{ such that } p_n \to 0 \text{ and } Tp_n \to q\}$.

Also, $S(T)$ is a closed linear subspace of Γ and moreover, by the closed Graph theorem, T is continuous if and only if $S(T) = \{0\}$. For a proof, see [2].

2. Jordan Banach Algebras

P. Jordan [4], a physicist, introduced Jordan algebras in an attempt to generalize a quantum mechanics formalism.

Definition 2.1. [6] A Jordan algebra Ω is a non associative algebra Ω whose product is non commutative and satisfies $(p.q).p^2 = p.(q.p^2), \forall p, q \in \Omega$. A Jordan Banach algebra Ω is a Jordan algebra endowed with a complete norm $||.||$ satisfying $||p.q|| \leq ||p||\, ||q||, \forall p, q \in \Omega$. Let us assume that Ω is unital with unit 1 and $||1|| = 1$.

If Ω is an algebra then Ω becomes a commutative Jordan algebra for the Jordan product $p \circ q = (pq + qp)/2$. If Ω is a Banach algebra (not of characteristic 2), then Ω becomes a commutative Jordan Banach algebra for the Jordan product $p \circ q = (pq + qp)/2$. Then, it is called a special Jordan Banach algebra, which is denoted by Ω^+. Here the Jordan product is commutative and distributive over addition.

An element p in a unital Jordan algebra Ω is said to be invertible if there exists q in Ω such that $p.q = 1$ and $p^2.q = p$. The spectrum $\sigma_\Omega(p)$ of an element $p \in \Omega$ is the set of all complex numbers λ such that $\lambda.1 - p$ is not invertible in Ω. By a theorem of Jacobson, the spectrum in the Jordan sense coincides with the classical spectrum in case of special Jordan algebra. The next result is also true, when spectral radius is defined in the usual sense.

Theorem 2.1. [1] *Let Ω be a Jordan Banach algebra and $p \in \Omega$. Then the spectral radius of p is $\lim_{n\to\infty} ||p^n||^{\frac{1}{n}}$.*

K. McCrimmon generalized the concept of Jacobson radical from associative algebras to Jordan algebras (see [7, 8]). An ideal I of a Jordan algebra Ω is said to be quasi-invertible, if for every $p \in I$, $1-p$ is invertible. For Ω, there exists a unique quasi-invertible ideal containing every quasi-invertible ideal, which is a result proved by K. McCrimmon. By definition, this ideal is the K. McCrimmon radical of Ω and it is also denoted by $Rad(\Omega)$. He proved that the K. McCrimmon radical coincides with the Jacobson radical in case of special Jordan algebra. If $Rad(\Omega) = \{0\}$ in a Jordan Banach algebra Ω, then Ω is said to be semisimple.

Theorem 2.2. [8] *Let Ω be an associative algebra. Then the McCrimmon radical $Rad(\Omega^+)$ of the Jordan algebra Ω^+ coincides with the Jacobson radical $rad(\Omega)$ of Ω. That is, $Rad(\Omega^+) = rad(\Omega)$.*

Definition 2.2. [5] Let Ω be a Banach algebra. A linear map $T : \Omega \to \Omega$ is called derivation, if $T(\rho.\eta) = \rho.T(\eta) + T(\rho).\eta, \forall \rho, \eta \in \Omega$.

Next, we introduce almost derivations on Banach algebras.

Definition 2.3. Let Ω be a Banach algebra. A linear map $T : \Omega \to \Omega$ is called almost derivation, if there exists $\epsilon \geq 0$ such that $||T(\rho.\eta) - \rho.T(\eta) - T(\rho).\eta|| \leq \epsilon||\rho||\ ||\eta||; \forall \rho, \eta \in \Omega$.

Remark 2.4. Let Ω be a Banach algebra. A linear map $T : \Omega \to \Omega$ is defined by $T(k) = \beta k$, $\forall k \in \Omega$ and for some $(\epsilon =)\beta \in (0, \infty)$. Then T is an almost derivation but not a derivation on Ω.

We can extend the concepts of derivation and almost derivation given in Definition 2.2 and Definition 2.3 as concepts of Jordan derivation and almost Jordan derivation on Jordan Banach algebras respectively.

A question form of a Kaplansky conjecture [5] of 1958 is the following. Is every derivation on semisimple Banach algebra continuous?. Kaplansky conjecture was proved by Johnson and Sinclair [3] in 1968. In 1996, Villena [13] proved that every derivation on a semisimple Jordan Banach algebra is continuous. There are some recent articles [9, 10, 11, 12] for automatic continuity of derivations in the theory of topological algebras.

Now, we extend Kaplansky conjecture as an open question for almost Jordan derivation on Jordan Banach algebras.

Problem 2.5. Let $T : \Omega \to \Omega$ be a almost Jordan derivation on a semisimple Jordan Banach algebra Ω. Is T continuous?

Also, we derive a partial solution to the open Problem 2.5. More specifically, we prove that every almost Jordan derivation T on a semisimple special Jordan Banach algebra Ω^+, with an additional condition on Ω^+, is continuous.

3. Main Result

In this section we assume that all Jordan Banach algebras are special Jordan Banach algebras.

Proposition 3.1. *Let Ω be a Banach algebra. Every almost derivation T on Ω is also a almost Jordan derivation on Ω^+, where Ω^+ is the special Jordan Banach algebra of Ω. This result is true even when multiplication in Ω is not associative.*

Proof. Since T is almost derivation, for $\rho, \eta \in \Omega$ we have

$$\begin{aligned}
& ||T(\rho \circ \eta) - T(\rho) \circ \eta - \rho \circ T(\eta)|| \\
= & ||T(\frac{1}{2}(\rho\eta + \eta\rho)) - \frac{1}{2}(T(\rho)\eta + \eta T(\rho)) - \frac{1}{2}(\rho T(\eta) + T(\eta)\rho)|| \\
= & \frac{1}{2}||T(\rho\eta) + T(\eta\rho) - T(\rho)\eta - \eta T(\rho) - \rho T(\eta) - T(\eta)\rho|| \\
\leq & \frac{1}{2}(||T(\rho\eta) - T(\rho)\eta - \rho T(\eta)|| + ||T(\eta\rho) - \eta T(\rho) - T(\eta)\rho||) \\
\leq & \frac{1}{2}(\epsilon||\rho||\,||\eta|| + \epsilon||\eta||\,||\rho||) \\
= & \epsilon||\rho||\,||\eta||.
\end{aligned}$$

So, we have $||T(\rho \circ \eta) - T(\rho) \circ \eta - \rho \circ T(\eta)|| \leq \epsilon||\rho||\,||\eta||$; for every $\rho, \eta \in \Omega$. □

Theorem 3.2. *Let Ω^+ be a special Jordan Banach algebra. If $T : \Omega^+ \to \Omega^+$ is a almost Jordan derivation, then the separating space $S(T)$ is a closed (two sided) ideal in Ω^+.*

Proof. Obviously $S(T)$ is a closed linear subspace of Ω^+.

Now, we prove that $S(T)$ is an ideal in Ω^+. Let $b \in S(T)$ and $c \in \Omega^+$. Then there exists a sequence $(a_n)_{n=1}^{\infty}$ in Ω^+ such that $a_n \to 0$, and $T(a_n) \to b$. Let $w = T(c)$. Also we have $c \circ a_n \to 0$. Since T is a almost Jordan derivation, we have

$$\begin{aligned}
||T(c \circ a_n) - c \circ b|| \leq & ||T(c \circ a_n) - c \circ T(a_n) - T(c) \circ a_n|| \\
& + ||c \circ T(a_n) + w \circ a_n - c \circ b|| \\
\leq & ||T(c \circ a_n) - c \circ T(a_n) - T(c) \circ a_n|| \\
& + ||c \circ T(a_n) - c \circ b|| + ||w \circ a_n|| \\
\leq & \epsilon||c||\,||a_n|| + ||c||\,||T(a_n) - b|| + ||w \circ a_n||.
\end{aligned}$$

Since $||T(a_n) - b|| \to 0$, $||a_n|| \to 0$ and $||w \circ a_n|| \to 0$ we have $||T(c \circ a_n) - c \circ b|| \to 0$, and hence $T(c \circ a_n) \to c \circ b$, when $c \circ a_n \to 0$. Therefore, we conclude that $c \circ b \in S(T)$. It was mentioned that the Jordan product $\circ$ is commutative. Hence $S(T)$ is a two sided ideal in Ω^+. □

Theorem 3.3. *Let Ω^+ be a special Jordan Banach algebra such that Ω^+ is semisimple, and r_{Ω^+} is continuous on Ω^+. If $T : \Omega^+ \to \Omega^+$ is a almost Jordan derivation with $r_{\Omega^+}(Ta) \leq ||a||, \forall a \in \Omega^+$. Then T is continuous.*

Proof. Let $b \in S(T)$. Then there exists $(a_n)_{n=1}^{\infty}$ in Ω^+ such that $a_n \to 0$ and $Ta_n \to b$. Since $r_{\Omega^+}(Ta) \leq ||a||$ and $||a_n|| \to 0$, we have $r_{\Omega^+}(Ta_n) \to 0$. Also, we have $r_{\Omega^+}(Ta_n) \to r_{\Omega^+}(b)$. So, we conclude that $r_{\Omega^+}(b) = 0$. By Theorem 3.2, $S(T)$ is an ideal in Ω^+. For every $c \in \Omega^+, b \circ c \in S(T)$. Therefore $r_{\Omega^+}(b \circ c) = 0$. By Theorem 2.2, $rad(\Omega^+) = \{c_1 \in \Omega^+ : r_{\Omega^+}(c_1 \circ c_2) = 0, \forall c_2 \in \Omega^+\}$, and hence $b \in rad(\Omega^+)$. So, $S(T) \subseteq rad(\Omega^+)$. Since Ω^+ is semisimple, $S(T) = \{0\}$. Therefore T is continuous, by the closed Graph theorem. □

From this Theorem 3.3, we get a partial solution to the open Problem 2.5 in case of special Jordan Banach algebras.

4. Conclusion

We extended the Kaplansky conjecture for Banach algebras to Jordan Banach algebras as an open Problem 2.5 to every almost Jordan derivation on a semisimple Jordan Banach algebra for automatic continuity. We obtained a partial solution to Problem 2.5 in Theorem 3.3.

Acknowledgment

The authors first like to thank the reviewers and the editor for their valuable comments and corrections for improvement of this article. The Council of Scientific and Industrial Research(CSIR) of India funded the work of the second author (Gurusamy Siva).

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Received : *December* 2021
Accepted : *January* 2022