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Multiplicity results for the nonlinear p-Laplacian fractional boundary value problems

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Abstract. This paper investigates the existence of a single and multiple positive solutions of fractional differential equations with p-Laplacian by means of the Green's function properties, the Guo-Krasnosel'skii fixed point theorem, the monotone iterative technique accompanied by established sufficient conditions and the Leggett-Williams fixed point theorem. Additionally, the main results are illustrated by some examples to show their validity.

1. Introduction

The extensive applications of fractional calculus has emerged in numerous fields of science and engineering which include blood flow phenomena, diffusive transport akin to diffusion, control theory of dynamical systems and continuum mechanics among others [7]-[12].

The study of boundary value problems (BVP) to nonlinear fractional differential equations has evidently proved to be inevitable as an enormous number of researchers are drawn to the investigation of the existence of positive solutions of BVPs for nonlinear fractional differential equations see [13], [27]-[28]. Existence of positive solutions for fractional BVP has expanded and consequently generated great results in both differential and integral boundary value problems [14]. As fractional differential equations are effective tools in the description of hereditary properties of various materials, fractional multi-point problems with non-resonance

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where considered [15]-[17], furthermore, fractional p-Laplace problems were covered in [6], [18]-[26].

As far as the authors are concerned, there are few papers that cover the existence of positive solutions of fractional differential equations with p-Laplacian and double multi point boundary values conditions. As a result, this work is crucial as it represents an advancement applicable in a vast range of fields with a greater degree of freedom. The study entailed herein is one of a kind and is motivated by the literature mentioned. In this paper we concentrate on the existence of positive solutions for a BVP of fractional differential equations

$$\begin{aligned} &D^{\beta}(\varphi_p(D^{\alpha}y(t))) + f(t, y(t)) = 0, \qquad t \in [0,1],\\ &y(0) = 0, \quad \varphi_p(D^{\alpha}y(0)) = 0,\\ &\varphi_p(D^{\alpha}y(1)) = \sum_{i=1}^{m-2} b_i\varphi_p(D^{\alpha}y(\xi_i)), \quad D^{\gamma}y(1) = \sum_{i=1}^{m-2} a_i D^{\gamma}y(\eta_i), \end{aligned} \tag{1}$$

where $1 < \alpha, \beta \leq 2,\ 0 < \gamma \leq 1$ such that $0 \leq \alpha - \gamma - 1,\ \ 0 \leq a_i, b_i, \eta_i, \xi_i \leq 1$, $i = 1, 2, \cdots, m-2,\ \sum_{i=1}^{m-2} a_i\eta_i < 1,\ \sum_{i=1}^{m-2} b_i\xi_i < 1,\ f \in ([0,1] \times [0,+\infty), [0,+\infty))$, $\varphi_p(s) = |s|^{p-2}s,\ p > 1,\ \varphi_p^{-1} = \varphi_q,\ \frac{1}{p} + \frac{1}{q} = 1$, with D^{α}, D^{β} and D^{γ} are the standard Riemann-Liouville fractional derivatives.

The paper is structured in such a manner, in Section 2, we will give some necessary definitions and lemmas which are used in the main results.We present the associated Green's function with its properties. For clarity, we also state some fixed point theorems. Section 3, deals with the existence of a single positive solution followed by a comprehensive example. In Section 4, we will give the multiplicity results for BVP (1). In the last parts of section 3 and 4, we come up with some examples to illustrate our main results.

2. Basic Definitions and Preliminaries

We first introduce some necessary definitions and lemmas in this section. The following auxiliary Lemmas are necessary to illustrate the existence of solutions for problem (1).

Definition 2.1. [1] The integral

$$I^{\beta}g(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds, \tag{2}$$

where $\beta > 0$, is the fractional integral of order β for a function $g(t)$.

Definition 2.2. [1] For a function $g(t)$ the expression

$$D_{0^+}^{\beta}g(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\beta-1} g(s)ds, \tag{3}$$

is called the Riemann-Liouville fractional derivative of order β, where $n = [\beta] + 1$, and $[\beta]$ denotes the integer part of number β.

DEFINITION 2.3. [5] The map θ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space K provided that $\theta : P \to [0, +\infty)$ is continuous and

$$\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y)$$

for all $x, y, \in P$ and $0 \leq t \leq 1$.

LEMMA 2.1. [1] *Assume that $g \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\beta > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I^\beta D^\beta g(t) = g(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_N t^{\beta-N}, \tag{4}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, N$, where N is the smallest integer greater than or equal to β.

LEMMA 2.2. [3] *Let $g \in C[0,1]$. Then the fractional differential equation*

$$D^\alpha y(t) + g(t) = 0$$

$$y(0) = 0, \qquad D^\gamma y(1) = \sum_{i=1}^{m-2} a_i D^\gamma y(\eta_i)$$

has a unique solution which is given by

$$y(t) = \int_0^1 G(t,s) g(s) ds,$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s),$$

in which

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{\sum_{0 \leq s \leq \eta_i} [a_i \eta_i^{\alpha-\gamma-1} t^{\alpha-1}(1-s)^{\alpha-\gamma-1} - a_i t^{\alpha-1}(\eta_i - s)^{\alpha-\gamma-1}]}{A\Gamma(\alpha)}, & t \in [0,1], \\ \frac{\sum_{\eta_i \leq s \leq 1} a_i \eta_i^{\alpha-\gamma-1} t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{A\Gamma(\alpha)}, & t \in [0,1], \end{cases} \tag{5}$$

where $A = 1 - \sum_{i=1}^{m-2} a_i \eta_i^{\alpha-\gamma-1}$.

LEMMA 2.3. *Let y be a continuous function. Then the linear fractional BVP*

$$D^{\beta}(\varphi_p(D^{\alpha}y(t))) + g(t) = 0, \qquad t \in [0,1],$$
$$y(0) = 0, \quad \varphi_p(D^{\alpha}y(0)) = 0,$$
$$\varphi_p(D^{\alpha}y(1)) = \sum_{i=1}^{m-2} b_i \varphi_p(D^{\alpha}y(\xi_i)), \quad D^{\gamma}y(1) = \sum_{i=1}^{m-2} a_i D^{\gamma}y(\eta_i)$$

has a unique solution given by

$$y(t) = \int_0^1 G(t,s)\rho(s)ds,$$

where

$$\rho(s) = \int_0^1 H(t,s)g(s)ds + \frac{t^{\beta-1}}{B}\sum_{i=1}^{m-2} b_i \int_0^1 H(\xi_i,s)g(s)ds,$$

in which

$$B = 1 - \sum_{i=1}^{m-2} b_i \xi_i^{\beta-1},$$

$$H(t,s) = \begin{cases} \frac{t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le t \le 1, \\ \frac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le t \le s \le 1, \end{cases}$$
$$\sum_{i=1}^{m-2} b_i H(\xi_i,s) = \begin{cases} \frac{\sum_{0\le s\le \xi_i}[b_i\xi_i^{\beta-1}(1-s)^{\beta-1} - b_i(\xi_i-s)^{\beta-1}]}{B\Gamma(\beta)}, & t \in [0,1], \\ \frac{\sum_{\xi_i \le s \le 1} b_i \xi_i^{\beta-1}(1-s)^{\beta-1}}{B\Gamma(\beta)}, & t \in [0,1]. \end{cases} \tag{6}$$

Proof. To simplify BVP (1), we let $D^{\alpha}y = w$, and $v = \varphi_p(w)$, so BVP (1) becomes the following linear BVP

$$D^{\beta}v(t) = g(t)$$
$$v(0) = 0 \quad \text{and} \quad v(1) = \sum_{i=1}^{m-2} b_i v(\xi_i), \tag{7}$$

where $g \in L'[0,1]$ and $g \ge 0$.
From Lemma 2.1 and problem (7), we get

$$v(t) = c_1 t^{\beta-1} + c_2 t^{\beta-2} + I^{\beta}g(t).$$

Since $v(0) = 0$, we have $c_2 = 0$ and so

$$v(t) = c_1 t^{\beta-1} - I^{\beta}g(t). \tag{8}$$

Considering the boundary condition in problem (7), $v(1) = \sum_{i=1}^{m-2} b_i v(\xi_i)$, we obtain

$$c_1 - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds = \sum_{i=1}^{m-2} b_i \left[c_1 \xi_i^{\beta-1} - \int_0^{\xi_i} \frac{(\xi_i - s)^{\beta-1}}{\Gamma(\beta)} g(s)ds \right]$$

$$c_1 \left[1 - \sum_{i=1}^{m-2} b_i \xi_i^{\beta-1} \right] = \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\beta-1}}{\Gamma(\beta)} g(s)ds$$

$$c_1 = \frac{1}{B\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} g(s)ds - \frac{1}{B\Gamma(\beta)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1} g(s)ds,$$

where $B = 1 - \sum_{i=1}^{m-2} b_i \xi_i^{\beta-1}$.
Substituting for c_1 into (8), we get

$$\begin{aligned}
v(t) =& \frac{t^{\beta-1}}{B\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} g(s)ds - \frac{t^{\beta-1}}{B\Gamma(\beta)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1} g(s)ds \\
& - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)ds \\
=& - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} g(s)ds \\
& + \frac{t^{\beta-1}}{B\Gamma(\beta)} \sum_{i=1}^{m-2} b_i \xi_i^{\beta-1} \int_0^1 (1-s)^{\beta-1} g(s)ds \\
& - \frac{t^{\beta-1}}{B\Gamma(\beta)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1} g(s)ds \\
=& - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} g(s)ds \\
& + \frac{t^{\beta-1}}{B\Gamma(\beta)} b_1 \xi_1^{\beta-1} \int_0^{\xi_1} (1-s)^{\beta-1} g(s)ds + \frac{t^{\beta-1}}{B\Gamma(\beta)} b_1 \xi_1^{\beta-1} \int_{\xi_1}^1 (1-s)^{\beta-1} g(s)ds \\
& - \frac{t^{\beta-1}}{B\Gamma(\beta)} b_1 \int_0^{\xi_1} (\xi_1 - s)^{\beta-1} g(s)ds + \cdots \\
& + \frac{t^{\beta-1}}{B\Gamma(\beta)} b_{m-2} \xi_{m-2}^{\beta-1} \int_0^{\xi_{m-2}} (1-s)^{\beta-1} g(s)ds \\
& + \frac{t^{\beta-1}}{B\Gamma(\beta)} b_{m-2} \xi_{m-2}^{\beta-1} \int_{\xi_{m-2}}^1 (1-s)^{\beta-1} g(s)ds
\end{aligned}$$

$$- \frac{t^{\beta-1}}{B\Gamma(\beta)} b_{m-2} \int_0^{\xi_{m-2}} (\xi_{m-2} - s)^{\beta-1} g(s) ds$$

$$= \int_0^1 H(t,s) g(s) ds + \frac{t^{\beta-1}}{B} \sum_{i=1}^{m-2} b_i \int_0^1 H(\xi_i, s) g(s) ds,$$

where

$$H(t,s) = \begin{cases} \frac{t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le t \le 1, \\ \frac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le t \le s \le 1, \end{cases}$$

$$\sum_{i=1}^{m-2} b_i H(\xi_i, s) = \begin{cases} \frac{\sum_{0 \le s \le \xi_i} [b_i \xi_i^{\beta-1}(1-s)^{\beta-1} - b_i(\xi_i - s)^{\beta-1}]}{B\Gamma(\beta)}, & t \in [0,1], \\ \frac{\sum_{\xi_i \le s \le 1} b_i \xi_i^{\beta-1}(1-s)^{\beta-1}}{B\Gamma(\beta)}, & t \in [0,1], \end{cases}$$

This completes the proof.

□

LEMMA 2.4. [2] *If* $\sum_{i=1}^{m-2} a_i \eta_i^{\alpha-\gamma-1} < 1$, *then the function* $G(t,s)$ *in* (5) *satisfies the following conditions:*

(1) $G(t,s) > 0$, *for* $s, t \in (0,1)$,
(2) $G(t,s) \le \overline{G}(t,s) \le G_*(s,s)$, *for* $s,t \in [0,1]$, *where*

$$\overline{G}(t,s) = \overline{G}_1(t,s) + \overline{G}_2(t,s),$$

in which

$$\overline{G}_1(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}$$

$$\overline{G}_2(t,s) = \frac{\sum_{i=1}^{m-2} a_i \eta_i^{\alpha-\gamma-1} t^{\alpha-1}(1-s)^{\alpha-1}}{A\Gamma(\alpha)},$$

$$G_*(s,s) = \max_{t\in[0,1]} \overline{G}_1(t,s) + \max_{t\in[0,1]} \overline{G}_2(t,s).$$

(3) $G(t,s) \ge t^{\alpha-1} G(1,s)$ *for all* $s,t \in [0,1]$.

LEMMA 2.5. *The function* $H(t,s)$ *defined by* (6) *respectively satisfy the following conditions:*

(1) $H(t,s) \ge 0$ *and* $H(t,s) \le H(s,s)$ *for* $s,t \in [0,1]$,
(2) *there exist a positive function* $g_2 \in C[0,1]$ *such that*

$$\min_{\vartheta \le t \le \delta} H(t,s) \ge g_2(s) H(s,s) \text{ for } s \in [0,1],$$

where

$$g_2(s) = \begin{cases} \frac{\delta^{\beta-1}(1-s)^{\beta-1} - (\delta-s)^{\beta-1}}{t^{\beta-1}(1-s)^{\beta-1}}, & \text{if } s \in [0, m_1], \\ (\frac{\vartheta}{s})^{\beta-1}, & \text{if } s \in [m_1, 1] \end{cases}$$

for $0 \leq \vartheta < m_1 < \delta \leq 1$,

(3) $\max_{0\leq t\leq 1} \int_0^1 H(t,s)ds = \frac{\Gamma(\beta)}{\Gamma(2\beta)}$.

PROOF. The proof will be given in three parts. By definition of $H(t,s)$, for all $(t,s) \in [0,1]\times[0,1]$ if $s \leq t$ it can be expressed as:

$$\begin{aligned} H(t,s) =& \frac{1}{\Gamma(\beta)}((t(1-s))^{\beta-1} - (t-s)^{\beta-1}) \\ \geq& \frac{t^{\beta-1}}{\Gamma(\beta)}((1-s)^{\beta-1} - (1-s)^{\beta-1}) \\ =& 0, \end{aligned}$$

if $t \leq s$, it can be easily be seen that $H(t,s) \geq 0$ for all $(t,s) \in [0,1]\times[0,1]$. Considering $H(t,s)$ for $s \leq t$, we define

$$L_H(t,s) = t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}$$

then

$$\begin{aligned} \frac{\partial L_H(t,s)}{\partial t} =& (\beta-1)t^{\beta-2}(1-s)^{\beta-1} - (\beta-1)(t-s)^{\beta-2} \\ \leq& (\beta-1)(1-s)^{\beta-1} - (\beta-1)(1-s)^{\beta-2} \\ \leq& (\beta-1)[(1-s)^{\beta-1} - (1-s)^{\beta-2}] \\ \leq& 0, \end{aligned}$$

which implies that $L_H(t,s)$ is non-increasing for all $s \in [0,1]$, therefore, we get

$$L_H(t,s) \leq L_H(s,s) \text{ for all } 0 \leq s \leq t \leq 1. \tag{9}$$

Then, by definition of H and (6), we obtain that $H(t,s) \leq H(s,s)$ for all $s,t \in [0,1]$.

We now let

$$J_H(t,s) = (t(1-s))^{\beta-1} \text{ for } t \leq s \leq 1.$$

We can see that $L_H(t,s)$ is non-increasing for $s \leq t$, and $J_H(t,s)$ to be non-decreasing for all $s \in [0,1]$ then

$$\begin{aligned} \min_{\vartheta\leq t\leq\delta} H(t,s) &= \frac{1}{\Gamma(\beta)}\begin{cases} L_H(\vartheta,s), & s \in [0,m_1], \\ J_H(\delta,s), & s \in [m_1,1], \end{cases} \\ &= \frac{1}{\Gamma(\beta)}\begin{cases} (\vartheta(1-s))^{\beta-1} - (\vartheta-s)^{\beta-1}, & s \in [0,m_1], \\ (\delta(1-s))^{\beta-1}, & s \in [m_1,1], \end{cases} \end{aligned}$$

for $\vartheta \leq m_1 \leq \delta$ satisfies the equation

$$(\vartheta(1-s))^{\beta-1} - (\vartheta-s)^{\beta-1} = (\delta(1-s))^{\beta-1}.$$

By the monotonicity of L_H and J_H, we have

$$\max_{0\le t\le 1} H(t,s)=H(s,s)=\frac{(s(1-s))^{\beta-1}}{\Gamma(\beta)}, \tag{10}$$

we assign $g_2(s)$ as stated from Lemma 2.5, it is clear that for $s\in[0,m_1]$, $s\le t$

$$\begin{aligned} g_2(s)H(s,s) &= \frac{((1-s)\delta)^{\beta-1}-(\delta-s)^{\beta-1}}{(t(1-s))^{\beta-1}}\times\frac{(s(1-s))^{\beta-1}}{\Gamma(\beta)} \\ &\le \frac{1}{\Gamma(\beta)}[((1-s)\delta)^{\beta-1}-(\delta-s)^{\beta-1}], \end{aligned}$$

since $g_2(s)$ is non-increasing, for $\vartheta\le\delta$, we get

$$g_2(s)H(s,s)\le\frac{((1-s)\vartheta)^{\beta-1}-(\vartheta-s)^{\beta-1}}{\Gamma(\beta)}.$$

Therefore,

$$\min_{\vartheta\le s,t\le\delta} H(t,s)\ge g_2(s)H(s,s).$$

Also, since $g_2(s)$ is non-decreasing for $s\in[m_1,1]$, $t\le s$ and $\vartheta\le\delta$,

$$\begin{aligned} g_2(s)H(s,s) &= \left(\frac{\vartheta}{s}\right)^{\beta-1}\times\frac{1}{\Gamma(\beta)}[s(1-s)]^{\beta-1} \\ &\le \frac{1}{\Gamma(\beta)}[(1-s)\vartheta]^{\beta-1} \\ &\le \frac{1}{\Gamma(\beta)}[(1-s)\delta]^{\beta-1}. \end{aligned}$$

Therefore,

$$\min_{\vartheta\le s,t\le\delta} H(t,s)\ge g_2(s)H(s,s) \text{ for all } s,t\in[0,1].$$

By the Beta integral function $B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}dt$, for $u,v\in\mathbb{R}$ and $B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$, using equation (10) we get

$$\begin{aligned} &\max_{0\le t\le 1}\left(\int_0^1 H(t,s)ds+\frac{t^{\beta-1}}{B}\sum_{i=1}^{m-2} b_i\int_0^1 H(\xi_i,s)ds\right) \\ &=\frac{1}{\Gamma(\beta)}\int_0^1 (s(1-s))^{\beta-1}ds+\frac{1}{B}\sum_{i=1}^{m-2} b_i\int_0^1 (s(1-s))^{\beta-1}ds \\ &=\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left[1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right]. \end{aligned} \tag{11}$$

Therefore,

$$\max_{0\le t\le 1}\int_0^1 H(t,s)ds=\frac{\Gamma(\beta)}{\Gamma(2\beta)}.$$

This completes the proof. □

Furthermore, we consider the following fixed point theorems and lemmas to show existence results.

THEOREM 2.6. [4] *Let K be a Banach space. $P \subseteq K$ be a cone, and Ω_1, Ω_2 be two bounded open balls of K centred at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous operator such that either*

(1) $\|Ty\| \leq \|y\|,\ y \in P \cap \partial\Omega_1$ *and* $\|Ty\| \geq \|y\|,\ y \in P \cap \partial\Omega_2$ *or*
(2) $\|Ty\| \geq \|y\|,\ y \in P \cap \partial\Omega_1$ *and* $\|Ty\| \leq \|y\|,\ y \in P \cap \partial\Omega_2$ *holds.*

Then T has a fixed point in $P \cap (\overline{\Omega}_2) \setminus \Omega_1$.

Let $a, b, c > 0$ be constants, $P_c = \{y \in P : \|y\| < c\}$,
$P(\theta, b, d) = \{y \in P : b \leq \theta(y), \|y\| \leq d\}$.

THEOREM 2.7. [5] *Let P be a cone in a real Banach space K.*
$P_c = \{x \in P | \|x\| \leq c\}$, θ be a nonnegative continuous concave functional on P such that $\theta(x) \leq \|x\|$ for all $x \in \overline{P}_c$ and
$P(\theta, b, d) = \{x \in P | b \leq \theta(x), \|x\| \leq d\}$. Suppose $B : \overline{P}_c \to \overline{P}_c$ is completely continuous and there exist constants $0 < a < b < d \leq c$ such that

(C_1) *$\{x \in P(\theta, b, d) | \theta(x) > b = \emptyset\}$ and $\theta(Bx) > b$ for $x \in P(\theta, b, d)$;*
(C_2) *$\|Bx\| < a$ for $x \leq a$;*
(C_3) *$\theta(Bx) > b$ for $x \in P(\theta, b, c)$ with $\|Bx\| > d$.*

Then B has at least three fixed points x_1, x_2 and x_3 with $\|x_1\| < a,\ b < \theta(x_2)$,
$a < \|x_3\|$ with $\theta(x_3) < b$.

Let $K = C[0, 1]$ be endowed with $\|y\| = \max_{0 \leq t \leq 1} |y(t)|$. We define the cone $P \subset K$ by $P = \{y \in K | y(t) \geq 0\}$. Let the nonnegative continuous concave functional θ on the cone P be defined by

$$\theta(y) = \min_{\vartheta \leq t \leq \delta} |y(t)|, \quad \text{where} \quad 0 < \vartheta < \delta < 1.$$

DEFINITION 2.4. A bounded linear operator T, acting from a Banach space X into another space Y, that transforms weakly-convergent sequences in X to norm-convergent sequences in Y. Equivalently, an operator T is completely-continuous if it maps every relatively weakly compact subset of X into a relatively compact subset of Y.

LEMMA 2.8. *Let $T : P \to K$ be the operator defined by*

$$(Ty)(t) := \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau + \frac{s^{\beta-1}}{B}\sum_{i=1}^{m-2} b_i \int_0^1 H(\xi_i,\tau)f(\tau,y(\tau))d\tau\right)ds. \tag{12}$$

Then $T : P \to P$ is completely continuous.

PROOF. Let $y \in P$, by the nonnegativeness and continuity of $G(t,s), H(t,s)$ and $f(t, y(t))$, we get $T : P \to P$ is continuous.
Let $\Omega \subset P$ be bounded, thus, there exists a positive constant $M > 0$ such that $\|y\| \leq M$ for all $y \in \Omega$. Let $L = \max_{0\leq t\leq 1, 0\leq y\leq M} |f(t,y)| + 1$, then for $y \in \Omega$, we get

$$\begin{aligned}
&|(Ty)(t)| = \\
&\left| \int_0^1 G(t,s)\varphi_q \left(\int_0^1 H(s,\tau) f(\tau, y(\tau)) d\tau + \frac{s^{\beta-1}}{B} \sum_{i=1}^{m-2} b_i \int_0^1 H(\xi_i, \tau) f(\tau, y(\tau)) d\tau \right) ds \right| \\
&\leq L^{q-1} \int_0^1 G(t,s)\varphi_q \left(\int_0^1 H(s,\tau) d\tau + \frac{s^{\beta-1}}{B} \sum_{i=1}^{m-2} b_i \int_0^1 H(\xi_i,\tau) d\tau \right) ds \\
&\leq \left[\frac{L\Gamma(\beta)}{\Gamma(2\beta)} \left(1 + \frac{1}{B} \sum_{i=1}^{m-2} b_i \right) \right]^{q-1} \int_0^1 G_*(s,s) ds, \\
&< +\infty.
\end{aligned}$$

which implies that $T(\Omega)$ is uniformly bounded.
Also, by the continuity of $G(t,s)$ and $H(t,s)$ on $[0,1] \times [0,1]$, we know that this is uniformly continuous on $[0,1] \times [0,1]$. Therefore, for fixed $s \in [0,1]$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$, such that $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$,

$$|G(t_1,s) - G(t_2,s)| < \varphi_p \left[\frac{L\Gamma(\beta)}{\Gamma(2\beta)} \left(1 + \frac{1}{B} \sum_{i=1}^{m-2} b_i \right) \right] \varepsilon.$$

Thus, for all $y \in \Omega$,

$$\begin{aligned}
|(Ty)(t_2) - (Ty)(t_1)| &\leq \int_0^1 |G(t_2,s) - G(t_1,s)| \left[\frac{L\Gamma(\beta)}{\Gamma(2\beta)} \left(1 + \frac{1}{B} \sum_{i=1}^{m-2} b_i \right) \right]^{q-1} ds \\
&\leq \varphi_q \left[\frac{L\Gamma(\beta)}{\Gamma(2\beta)} \left(1 + \frac{1}{B} \sum_{i=1}^{m-2} b_i \right) \right] \int_0^1 |G(t_2,s) - G(t_1,s)| ds \\
&\leq \varepsilon,
\end{aligned}$$

which means that $T(\Omega)$ is equicontinuous and by the Arzella-Ascoli theorem, we obtain $T : P \to P$ is completely continuous. □

3. Existence of a single positive solution for BVP (1)

For convenience sake, we denote

$$\mathcal{M} = \left[\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)} \left(1 + \frac{1}{B} \sum_{i=1}^{m-2} b_i \right) \right]^{q-1} \int_0^1 G_*(s,s)ds \right]^{-1},$$

$$\mathcal{N} = \left[\int_\vartheta^\delta t^{\alpha-1} G(1,s) \varphi_q \left(\int_\vartheta^\delta g_2(\tau) H(\tau,\tau) \left(1 + \frac{1}{B} \sum_{i=1}^{m-2} b_i \right) d\tau \right) ds \right]^{-1}.$$

THEOREM 3.1. *Let $f(t,y)$ be continuous on $[0,1] \times [0,+\infty)$. Assume that there exist two positive constants $a_2 > a_1 > 0$ such that*

(A_1) $f(t,y) \geq \varphi_p(\mathcal{N} a_1)$ *for* $(t,y) \in [0,1] \times [0,a_1]$;
(A_2) $f(t,y) \leq \varphi_p(\mathcal{M} a_2)$ *for* $(t,y) \in [0,1] \times [0,a_2]$.

Then the fractional differential equation boundary value problem (1) has at least one positive solution y such that $a_1 \leq \|y\| \leq a_2$.

PROOF. By Lemma 2.8, we can ascertain that $T: P \to P$ is completely continuous and the fractional differential equation BVP (1) has a solution $y = y(t)$ if and only if y solves the operator equation $y = Ty(t)$.

The proof is presented in two steps.

Step 1: Let $\Omega_1 := \{y \in P | \|y\| < a_1\}$. For $y \in \partial\Omega_1$ we get $0 \leq y(t) \leq a_1$ for all $t \in [0,1]$. It follows from (A_1) that $t \in [\vartheta, \delta]$,

$$\begin{aligned}
&(Ty)(t) = \\
&\int_0^1 G(t,s)\varphi_q \left(\int_0^1 H(s,\tau) f(\tau, y(\tau)) d\tau + \frac{s^{\beta-1}}{B} \sum_{i=1}^{m-2} b_i \int_0^1 H(\xi_i,\tau) f(\tau,y(\tau)) d\tau \right) ds \\
&\geq \mathcal{N} a_1 \int_0^1 t^{\alpha-1} G(1,s) \varphi_q \left(\int_\vartheta^\delta g_2(\tau) H(\tau,\tau) \left(1 + \frac{1}{B} \sum_{i=1}^{m-2} b_i \right) d\tau \right) ds \\
&\geq \mathcal{N} a_1 \int_\vartheta^\delta t^{\alpha-1} G(1,s) \varphi_q \left(\int_\vartheta^\delta g_2(\tau) H(\tau,\tau) \left(1 + \frac{1}{B} \sum_{i=1}^{m-2} b_i \right) d\tau \right) ds \\
&= a_1 = \|y\|.
\end{aligned}$$

Therefore,

$$\|Ty\| \geq \|y\| \quad \text{for} \quad y \in \partial\Omega_1.$$

Step 2: Let $\Omega_2 := \{y \in P | \|y\| < a_2\}$. For $y \in \partial\Omega_2$, we get $0 \leq y(t) \leq a_2$ for all $t \in [0,1]$. It follows from (A_2) that for $t \in [0,1]$.

$$\begin{aligned}
&\|Ty(t)\| = \max_{0\leq t\leq 1} \\
&\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau + \frac{s^{\beta-1}}{B}\sum_{i=1}^{m-2} b_i \int_0^1 H(\xi_i,\tau)f(\tau,y(\tau))d\tau\right) ds \\
&\leq \mathcal{M}a_2\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right)\right]^{q-1}\int_0^1 G_*(s,s)ds \\
&= a_2 = \|y\|.
\end{aligned}$$

Thus,

$$\|Ty\| \leq \|y\| \quad \text{for} \quad y \in \partial\Omega_2.$$

Then, by Theorem 2.6, this completes the proof. □

EXAMPLE 3.1. Consider the following boundary value problem:

$$\begin{aligned}
&D^\beta(\varphi_p(D^\alpha y(t))) + f(t,y(t)) = 0, \qquad t \in [0,1], \\
&y(0) = 0, \quad \varphi_p(D^\alpha y(0)) = 0, \\
&\varphi_p(D^\alpha y(1)) = \sum_{i=1}^{m-2} a_i\varphi_p(D^\alpha y(\xi_i)), \quad D^\gamma y(1) = \sum_{i=1}^{m-2} b_i D^\gamma y(\eta_i),
\end{aligned} \tag{13}$$

where

$$f(t,y(t)) = \frac{1}{100}\left(135 + y^{\frac{1}{100}} + 2t\right),$$

$\alpha = \frac{3}{2},\ \beta = \frac{5}{4},\ \gamma = \frac{1}{2},\ p = q = 2,\ m = 4,\ a_1 = b_1 = \frac{1}{2},\ a_2 = b_2 = \frac{1}{5},\ \xi_1 = \frac{1}{8},$
$\eta_1 = \frac{1}{9}, \xi_2 = \eta_2 = \frac{1}{3},$
and $f \in C([0,1] \times [0,+\infty),\ [0,+\infty))$.

We set $\vartheta = \frac{1}{3}$ and $\delta = \frac{2}{3}$. By computation we see that $A = \frac{3}{10}$, $B = 0.57666$,

$$\begin{aligned}
\mathcal{N} &= \left[\int_{\vartheta}^{\delta} G(1,s)\varphi_q\left(\int_{\vartheta}^{\delta} g_2(\tau)H(\tau,\tau)\left(1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right)d\tau\right)ds\right]^{-1}\\
&= \Bigg[\left(\int_{\vartheta}^{\delta} G_1(1,s)ds + \int_{\vartheta}^{\delta} G_2(1,s)ds\right)\\
&\quad \left(\frac{1}{\Gamma(\beta)}\int_{\vartheta}^{\delta} g_2(\tau)(\tau(1-\tau))^{\beta-1}\left(1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right)d\tau\right)^{q-1}\Bigg]^{-1}\\
&= \Bigg[\Bigg(\frac{1}{\Gamma(\alpha)}\int_{\vartheta}^{\delta} 1-(1-s)^{\alpha-1}ds + \frac{a_1}{A\Gamma(\alpha)}\int_0^{\eta_1} \eta_1^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1}\\
&\quad -(\eta_1-s)^{\alpha-\gamma-1}ds + \frac{a_1}{A\Gamma(\alpha)}\int_{\eta_1}^{\delta} \eta_1^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1}ds\\
&\quad +\frac{a_2}{A\Gamma(\alpha)}\int_0^{\eta_2} \eta_2^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1} - (\eta_2-s)^{\alpha-\gamma-1}ds\\
&\quad +\frac{a_2}{A\Gamma(\alpha)}\int_{\eta_2}^{\delta} \eta_2^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1}ds\Bigg)\\
&\quad \times\left(\frac{1}{\Gamma(\beta)}\int_{\vartheta}^{\delta} g_2(\tau)(\tau(1-\tau))^{\beta-1}\left(1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right)d\tau\right)^{q-1}\Bigg]^{-1}\\
&=1.3368
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M} &= \left[\int_0^1 G_*(s,s)ds\left(\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right)\right)^{q-1}\right]^{-1}\\
&= \Bigg[\left(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\gamma-1}ds + \frac{\sum_{i=1}^{2} a_i}{A\Gamma(\alpha)}\int_0^1 \eta_i^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1}ds\right)\\
&\quad \times\left(\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right)\right)^{q-1}\Bigg]^{-1}.\\
&=0.0858.
\end{aligned}$$

We set $a_1 = 1$ and $a_2 = 17$, therefore

$$f(t,y) = \frac{1}{100}\left(135 + y^{\frac{1}{100}} + 2t\right) \geq 1.35 > \varphi_p(\mathcal{N}a_1) \approx 1.3368$$
for $(t,y) \in [0,1] \times [0,1]$,
$$f(t,y) = \frac{1}{100}\left(135 + y^{\frac{1}{100}} + 2t\right) \leq 1.3803 < \varphi_p(\mathcal{M}a_2) = 1.4586$$
for $(t,y) \in [0,1] \times [0,17]$.

By Theorem 3.1, the fractional differential equation BVP (13) has at least one solution y such that $1 \leq \|y\| \leq 17$.

4. Existence of multiple positive solution

Assume the following hold:

(H_1) $f : [0,1] \times [0,+\infty) \to (0,+\infty)$ is continuous and nondecreasing, and there exists a constant $\gamma_1 > 0$ such that, for any $t \in [0,1], y \in [0,+\infty)$,

$$f(t, c_1 y) \geq c_1^{\gamma_1} f(t,y) \text{ for } 0 < c_1 \leq 1. \tag{14}$$

Remark 1. By (14), for any $c_1 \geq 1, (t,y) \in [0,1] \times [0,+\infty)$, it is clear that

$$f(t, c_1 y) \leq c_1^{\gamma_1} f(t,y).$$

Since T is completely continuous by Lemma 2.8, we also notice the monotonicity of f on y and the definition of T, it is easy to see that the operator T is nondecreasing. We define

$$l = \max_{t\in[0,1]} f(t,1). \tag{15}$$

Theorem 4.1. *Suppose condition (H_1) hold. If there exists a positive constant $b > 1$ such that*

$$\frac{l^{q-1}}{\mathcal{M}} \leq b_1^{1-\gamma_1(q-1)}, \tag{16}$$

where l is defined by (15), then the BVP (1) has the maximal and minimal solutions v^* and w^*, which are positive, and there exist two positive constants $m_1 \leq m_2$ such that

$$\begin{aligned} m_2 g_1(t) &\leq v^*(t) \leq b_1, \\ m_1 g_1(t) &\leq w^*(t) \leq b_1, \quad t \in [0,1], \end{aligned}$$

where

$$g_1(t) = t^{\alpha-1}.$$

Furthermore, for initial values $v_0^* = b_1$ and $y_0^* = 0$, we define the iterative sequences v_n^* and y_n^* by

$$v_n^*(t) = (Tv_{n-1}^*)(t) = T^n v_0^*(t),$$
$$w_n^*(t) = (Tw_{n-1}^*)(t) = T^n w_0^*(t).$$

Then,

$$\lim_{n\to+\infty} v_n^* = \overline{v}^*, \quad \lim_{n\to+\infty} w_n^* = \overline{w}^*$$

for $t \in [0,1]$ uniformly, respectively.

Proof. Let $B_{b_1} = \{y \in P : 0 \leq \|y\| \leq b_1\}$; we prove $T(B_{b_1}) \subset B_{b_1}$. Since for any $y \in B_{b_1}$, we have

$$0 \leq y(t) \leq \max_{t\in[0,1]} y(t) = \|y\| \leq b_1.$$

By (H_1), we get

$$\begin{aligned} 0 \leq f(t, y(t)) \leq & f(t, b_1) \\ & \leq b_1^{\gamma_1} f(t,1) \leq b_1^{\gamma_1} \max_{t\in[0,1]} f(t,1) = l b_1^{\gamma_1}. \end{aligned}$$

It follows from Lemma 2.8 that $T : P \to P$ is completely continuous operator, therefore by (16) and (19), we get

$$\begin{aligned} \|Ty(t)\| &= \max_{0\leq t\leq 1} \int_0^1 G(t,s)\varphi_q \\ & \left(\int_0^1 H(s,\tau) f(\tau, y(\tau)) d\tau + \frac{s^{\beta-1}}{B} \sum_{i=1}^{m-2} b_i \int_0^1 H(\xi_i,\tau) f(\tau,y(\tau)) d\tau \right) ds \\ &\leq (l b_1^{\gamma_1})^{q-1} \left[\frac{\Gamma(\beta)}{\Gamma(2\beta)} \left(1 + \frac{1}{B}\sum_{i=1}^{m-2} b_i \right) \right]^{q-1} \int_0^1 G_*(s,s) ds \\ &= \frac{(l b_1^{\gamma_1})^{q-1}}{\mathcal{M}} \\ &\leq b_1, \end{aligned}$$

which implies that $T(B_{b_1}) \subset B_{b_1}$.
Let $w_0^*(t) = 0, t \in [0,1]$, then $w_0^*(t) \in B_{b_1}$. We let $w_1^*(t) = (Tw_0^*)(t)$, we get $w_1^* \in B_{b_1}$. We denote

$$w_{n+1}^* = Tw_n^* = T^{n+1} w_0^*, \; n = 1, 2, \cdots.$$

It follows from $T(B_{b_1}) \subset B_{b_1}$that $w_n^* \in B_{b_1}$. Since T is compact, we have $\{w_n^*\}$ is sequentially compact set. By $w_1^* = Tw_0^* = T0 \in B_{b_1}$, we get

$$\begin{aligned} w_1^*(t) =& (Tw_0^*)(t) \\ =& (T0)(t) \geq 0 = w_0^*(t), \quad t \in [0,1]. \end{aligned}$$

By induction, we have

$$w_{n+1}^* \geq w_n^*,\ n = 0, 1, 2, \cdots .$$

As a result, there exists $\overline{w}^* \in B_{b_1}$ such that $w_n^* \to \overline{w}^*$. We let $n \to +\infty$, from the continuity of T and $Tw_n^* = w_{n-1}^*$, we get $T\overline{w}^* = \overline{w}^*$, which implies that $\overline{w}^*$ is a positive solution of BVP (1). Since $f : [0,1] \times [0,\infty) \to (0,+\infty)$ it is evident that the zero function is not the solution of BVP (1), therefore, $\max_{0 \leq t \leq 1} |\overline{w}^*(t)| > 0$; by $\overline{w}^* \in P$, we get

$$\overline{w}^*(t) \geq \|\overline{w}^*\| g_1(t) > 0,\ t \in (0,1), \tag{17}$$

thus, $\overline{w}^*(t)$ is a positive solution of BVP (1).
Conversely, let $v_0^*(t) = b_1, t \in [0,1]$, then $v_0^*(t) \in B_{b_1}$. We let $v_1^* = Tv_0^*$, clearly we have $v_1^* \in B_{b_1}$. We denote

$$v_{n+1}^* = Tv_n^* = T^{n+1}v_0^*,\ n = 1, 2, \cdots .$$

It follows from $T(B_{b_1}) \subset B_{b_1}$ that

$$v_n^* \in B_{b_1}, \quad n = 0, 1, 2, \cdots . \tag{18}$$

Since T is compact by Lemma 2.8, we can see that $\{v_n^*\}$ is a sequentially compact set. Since $v_1^* \in B_{b_1}$, we have

$$0 \leq v_1^*(t) \leq \|v_1^*\| \leq b = v_0^*(t).$$

If follows from Lemma 2.8 that $T : P \to P$ is nondecreasing, therefore

$$v_2^* = Tv_1^* \leq Tv_0^* = v_1^*.$$

As a result, there exists $\overline{v}^* \in B_{b_1}$ such that $v_n^* \to \overline{v}^*$. We let $n \to +\infty$, from the continuity of T and $Tv_n^* = v_{n-1}^*$, we get $T\overline{v}^* = \overline{v}^*$, which implies that $\overline{v}^*$ is a nonnegative solution of BVP (1).
We note that $v_0^* = b_1 w_0^* = 0, t \in [0,1]$, therefore it follows from monotonicity of T that $Tv_0^* \geq Tw_0^*$: by induction, we get $v_n^* \geq w_n^*, n = 0, 1, 2, \cdots$, which implies that $\overline{v}^* \geq \overline{w}^*$. therefore by (17) we get

$$\overline{v}^* \geq \overline{w}^* \geq \|\overline{v}^*\| g_1(t) > 0,\ t \in (0,1).$$

This implies that $\overline{v}^*$ is also a positive solution of BVP (1). Finally, we let u^* be any fixed point of T in B_{b_1}, then

$$w_0^* = 0 \leq u^* \leq b_1 = v_0^*,$$

and then

$$w_1^* = Tw_0^* \leq Tu^* = u^* \leq Tb_1 = v_1^*.$$

By induction we get

$$w_n^* \leq u^* \leq v_n^*,\ n = 0, 1, 2, \cdots .$$

This implies that $\overline{v}^*$ and $\overline{w}^*$ are maximal and minimal solutions of the BVP (1). Let $m_1 = \|\overline{w}^*\|, m_2 = \|\overline{v}^*\|$, then we get

$$m_2 g_1(t) \leq \overline{v}^*(t) \leq b_1,$$
$$m_1 g_1(t) \leq \overline{w}^*(t) \leq b_1, \ t \in [0,1].$$

This completes the proof. □

Corollary 4.1.1. *Suppose condition (H_1) holds. If*

$$\gamma_1 < p - 1. \tag{19}$$

Then there exists a constant $b_1 > 1$ such that BVP (1) has the maximal and minimal solutions $\overline{v}^*$ and $\overline{w}^*$, which are positive, and there exist two positive constants $m_1 \leq m_2$ such that

$$m_2 g_1(t) \leq \overline{v}^*(t) \leq b_1,$$
$$m_1 g_1(t) \leq \overline{w}^*(t) \leq b_1, \ t \in [0,1].$$

Furthermore, for initial values $v_0^* = b_1$ and $w_0^* = 0$, we define the iterative sequences v_n^* and w_n^* by

$$v_n^*(t) = (Tv_{n-1}^*)(t) = T^n v_0^*,$$
$$w_n^*(t) = (Tw_{n-1}^*)(t) = T^n w_0^*.$$

Then

$$\lim_{n\to+\infty} v_n^* = \overline{v}^*, \quad \lim_{n\to+\infty} w_n^* = \overline{w}^*$$

for $t \in [0,1]$ uniformly, respectively.

Proof. It follows from $\gamma_1 < p - 1$ that

$$\lim_{u\to+\infty} \frac{u^{\gamma_1}}{u^{p-1}} = 0,$$

which implies that there exists $b > 2$ sufficiently large such that

$$\frac{b_1^{\gamma_1}}{b_1^{p-1}} < \frac{\mathcal{M}^{p-1}}{l} \tag{20}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, (20) becomes

$$\frac{l^{q-1}}{\mathcal{M}} \leq b_1^{1-\gamma_1(q-1)}.$$

By Theorem 4.1, the conclusion of Corollary 4.1.1 holds. □

Remark 2. In Corollary 4.1.1, we ascertained that (1) has maximal and minimal solutions $\overline{v}^*$ and $\overline{w}^*$ only by comparing $p-1$ to γ_1. Thus, (19) is satisfied.

EXAMPLE 4.1. Consider the following boundary value problem:

$$\begin{aligned}
&D^{\beta}(\varphi_p(D^{\alpha}y(t)))+f(t,y(t))=0, \qquad t\in[0,1],\\
&y(0)=0, \quad \varphi_p(D^{\alpha}y(0))=0,\\
&\varphi_p(D^{\alpha}y(1))=\sum_{i=1}^{m-2}a_i\varphi_p(D^{\alpha}y(\xi_i)), \quad D^{\gamma}y(1)=\sum_{i=1}^{m-2}b_iD^{\gamma}y(\eta_i),
\end{aligned} \tag{21}$$

where

$$f(t,y(t))=\sin ty(t)+t^2y^{\frac{1}{2}}(t).$$

$\alpha=\frac{3}{2},\ \beta=\frac{5}{4},\ \gamma=\frac{1}{2},\ p=3,\ m=4,\ a_1=b_1=\frac{1}{2},\ a_2=b_2=\frac{1}{5},\ \xi_1=\frac{1}{8},$
$\eta_1=\frac{1}{9}, \xi_2=\eta_2=\frac{1}{3},$
and $f\in C([0,1]\times[0,+\infty),\ [0,+\infty))$.
For any $0<c\le 1$ and $y\in[0,+\infty)$, we get

$$\begin{aligned}
f(t,cy)=&\sin t(cy)+t^2(cy)^{\frac{1}{2}}\\
\ge&\sin t(cy)+t^2c(y)^{\frac{1}{2}}\\
\ge&cf(t,y).
\end{aligned}$$

Setting $\gamma_1=1$, then

$$\gamma_1=1<p-1=2,$$

which implies that (4) holds. By Corollary 4.1.1, BVP (21) has at least two positive solutions.

THEOREM 4.2. *Let $f(t,y)$ be continuous on $[0,1]\times[0,+\infty)$. Assume that there exist constants $0<a<b<c$ such that the following assumptions hold;*

(B_1) *$f(t,y)\le\varphi_p(\mathcal{M}a)$ for $(t,y)\in[0,1]\times[0,a]$;*
(B_2) *$f(t,y)\ge\varphi_p(\mathcal{N}b)$ for $(t,y)\in[\vartheta,\delta]\times[b,c]$;*
(B_3) *$f(t,y)\le\varphi_p(\mathcal{M}c)$ for $(t,y)\in[0,1]\times[0,c]$.*

Then the fractional differential equation boundary value problem (1) has at least three positive solutions y_1, y_2 and y_3 with

$$\begin{aligned}
&\max_{0\le t\le 1}|y_1(t)|<a, \qquad b<\min_{\vartheta\le t\le\delta}|y_2(t)|<\max_{0\le t\le 1}|y_2(t)|\le c,\\
&a<\max_{0\le t\le 1}|y_3(t)|\le c, \qquad \min_{\vartheta\le t\le\delta}|y_3(t)|<b.
\end{aligned}$$

PROOF. By Lemma 2.8, we get that $T:P\to P$ is completely continuous and the fractional differential equation BVP (1) has a solution $y=y(t)$ if and only if y satisfies the operator equation $y=Ty(t)$.
We ascertain that all conditions of Theorem 2.7 are satisfied. If $y\in\overline{P}_c$, then $\|y\|<c$.

By (B_3), we have

$$\begin{aligned}
\|Ty(t)\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)\varphi_q \\
&\quad \left(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau + \frac{s^{\beta-1}}{B}\sum_{i=1}^{m-2} b_i \int_0^1 H(\xi_i,\tau)f(\tau,y(\tau))d\tau\right) ds \\
&\le \mathcal{M}c\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right)\right]^{q-1}\int_0^1 G_*(s,s)ds \\
&= c.
\end{aligned}$$

Thus, $T:\overline{P}_c \to \overline{P}_c$. Similarly, if $y\in\overline{P}_a$, then assumption (B_1) yields $\|Ty\|<a$. Hence, condition (C_2) of Theorem 2.7 is satisfied. To verify condition (C_1), we choose $y(t)=\frac{(b+c)}{2}, 0\le t\le 1$. It is obvious that $y(t)=\frac{(b+c)}{2}\in P(\theta,b,c)$ then $b\le y(t)\le c$ for $\vartheta\le t\le\delta$. Thus,

$$\begin{aligned}
\theta(Ty) &= \min_{\vartheta\le t\le\delta}|Ty(t)| \\
&\ge \mathcal{N}b\int_0^1 t^{\alpha-1}G(1,s)\varphi_q\left(\int_0^1 g_2(\tau)H(\tau,\tau)\left(1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right)d\tau\right)ds \\
&\ge \mathcal{N}b\int_\vartheta^\delta t^{\alpha-1}G(1,s)\varphi_q\left(\int_\vartheta^\delta g_2(\tau)H(\tau,\tau)\left(1+\frac{1}{B}\sum_{i=1}^{m-2} b_i\right)d\tau\right)ds \\
&= b,
\end{aligned}$$

therefore, $\theta(Ty)>b$ for all $y\in P(\theta,b,c)$. We set $d=c$, this implies that condition (C_1) of Theorem 2.7 is satisfied.
Similarly, if $y\in P(\theta,b,c)$ and $\|Ty\|>c=d$, we obtain $\theta(Ty)>b$. Then condition (C_3) of Theorem 2.7 is also satisfied. From Theorem 2.7, the fractional differential equation BVP (1) has at least three positive solutions y_1, y_2 and y_3, satisfying

$$\begin{aligned}
&\max_{0\le t\le 1}|y_1(t)|<a, \qquad b<\min_{\vartheta\le t\le\delta}|y_2(t)|, \\
&a<\max_{0\le t\le 1}|y_3(t)|, \qquad \min_{\vartheta\le t\le\delta}|y_3(t)|<b.
\end{aligned}$$

This completes the proof. □

EXAMPLE 4.2. Consider the following boundary value problem:

$$\begin{aligned}
&D^\beta(\varphi_p(D^\alpha y(t)))+f(t,y(t))=0, \qquad t\in[0,1], \\
&y(0)=0, \quad \varphi_p(D^\alpha y(0))=0, \\
&\varphi_p(D^\alpha y(1))=\sum_{i=1}^{m-2} a_i\varphi_p(D^\alpha y(\xi_i)), \quad D^\gamma y(1)=\sum_{i=1}^{m-2} b_i D^\gamma y(\eta_i),
\end{aligned} \tag{22}$$

where

$$f(t,y(t)) = \begin{cases} \frac{27}{20}y^3 + \frac{t}{1000} \text{ for } y \leq 1, \\ \frac{27}{20}y^{\frac{2}{5}} + \frac{t}{1000} \text{ for } y \geq 1, \end{cases}$$

$\alpha = \frac{3}{2},\ \beta = \frac{5}{4},\ \gamma = \frac{1}{2},\ p = q = 2,\ m = 4,\ a_1 = b_1 = \frac{1}{2},\ a_2 = b_2 = \frac{1}{5},\ \xi_1 = \frac{1}{8},$
$\eta_1 = \frac{1}{9}, \xi_2 = \eta_2 = \frac{1}{3},$
and $f \in C([0,1] \times [0,+\infty),\ [0,+\infty))$.
We set $\vartheta = \frac{1}{3}, \delta = \frac{2}{3}, a = 0.23, b = 1$ and $C = 100$. By computation, we see that $A = \frac{3}{10}$,
$B = 0.57666, \mathcal{N} = 1.3368$ and $\mathcal{M} = 0.0858$. Thus,

$$f(t,y) = \frac{27}{20}y^3 + \frac{t}{1000} \leq 0.0174 < \varphi_2(\mathcal{M}a) = 0.0197 \quad \text{for } (t,y) \in [0,1] \times [0,0.23],$$
$$f(t,y) = \frac{27}{20}y^{\frac{2}{5}} + \frac{t}{1000} \geq 1.35 > \varphi_2(\mathcal{N}b) \approx 1.3368 \text{ for } (t,y) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times [1,100],$$
$$f(t,y) = \frac{27}{20}y^{\frac{2}{5}} + \frac{t}{1000} \leq 8.5190 < \varphi_2(\mathcal{M}c) = 8.58 \text{ for } (t,y) \in [0,1] \times [0,100].$$

By Theorem 4.2, the fractional differential equation BVP (22) has at least three positive solutions y_1, y_2 and y_3 with

$$\max_{0 \leq t \leq 1} |y_1(t)| < 0.23, \qquad 1 < \min_{\frac{1}{3} \leq t \leq \frac{2}{3}} |y_2(t)| < \max_{0 \leq t \leq 1} |y_2(t)| \leq 100,$$
$$0.23 < \max_{0 \leq t \leq 1} |y_3(t)| \leq 100, \qquad \min_{\frac{1}{3} \leq t \leq \frac{2}{3}} |y_3(t)| < 1.$$

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