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Best simultaneous approximation in $L^p(S, X)$

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Abstract. As a counterpart to best approximation in normed linear spaces, best simultaneous approximation was introduced. In this paper, we shall consider relation between simultaneous proximinality W in X and $L^p(S, W)$ in $L^p(S, X)$ for $1 \leq p \leq \infty$. Also we consider relation between w-simultaneous proximinality W in X and $L^p(S, W)$ in $L^p(S, X)$ for $1 \leq p \leq \infty$.

1. Introduction

Let X be a normed linear space and W a nonempty subset of X. Then a point $w_0 \in W$ is said to be a best approximation for $x \in X$ if for every $w \in W$,

$$\|x - w_0\| \leq \|x - w\|.$$

If every $x \in X$ has at least one best approximation in W, then W is called a proximinal subset of X. If every $x \in X$ has a unique best approximation in W, then W is called a Chebyshev subset of X.
Also if $x \in X$ extend to bounded set $C \subseteq X$, we have following definition.

Definition 1.1. Let W be a subspace of X and C a bounded set in X. Then a point $w_0 \in W$ is said to be a best simultaneous approximation for C from W if

$$d(C, W) = \sup_{c \in C} \|c - w_0\|$$

where

$$d(C, W) = \inf_{w \in W} \sup_{c \in C} \|c - w\|.$$

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Let X be a Banach space and $(S, \mathcal{A}, \mu)$ a finite measure space. A function $\varphi : S \to X$ is said to be simple if its range contains only finitely many points $x_1, x_2, \ldots, x_n \in X$, and if $\varphi^{-1}(x_i)$ is measurable for all $i = 1, 2, \ldots, n$. Such φ can be written as $\varphi = \sum_{i=1}^{n} x_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of the set $E_i = \varphi^{-1}(x_i)$. A function $f : S \to X$ is said to be strongly measurable if there exists a sequence $\{\varphi_n\}$ of simple functions with $\lim_{n\to\infty} \|\varphi_n(t) - f(t)\| = 0$ almost everywhere $t \in S$.

The space of Bochner p-integrable functions is denoted by $L^p(S, X)$ which contains of all strongly measurable functions $f : S \to X$ such that

$$\int_S \|f(t)\|^p d\mu(t) < \infty \quad , \quad 1 \leq p < \infty \cdot$$

The norm in $L^p(S, X)$ is defined to be $\|f\|_p = (\int_S \|f(t)\|^p d\mu(t))^{\frac{1}{p}}$. It is known that $L^p(S, X)$ is a Banach space. It is clear that if W is a closed subspace of a Banach space X, then $L^p(S, W)$ is a closed subspace of $L^p(S, X)$, $1 \leq p < \infty$. In the following we give some lemmas that are need for main results.

Lemma 1.1. [2] *Let X be a Banach space and $1 < p < \infty$. Then X is reflexive if and only if $L^p(S, X)$ is reflexive.*

Lemma 1.2. [2] *If X is a uniformly convex space, then $L^p(S, X)$ for $1 < p < \infty$ is a uniformly convex space.*

Lemma 1.3. [4] *Let W be a closed subspace of X. Then $g \in L^1(S, W)$ is a best approximation for an element f of $L^1(S, X)$ if and only if for almost all $s \in S$, $g(s)$ is a best approximation for $f(s)$.*

2. Main Results

In this section we give characterizations of best simultaneous approximinality in $L^p(S, X)$.

If for every bounded set C there is at least one best simultaneous approximation in W, then W is called a *simultaneous proximinal subspace* of X. If for every bounded set C there is a unique best simultaneous approximation in W, then W is called a *simultaneous Chebyshev subspace* of X.

Let W be a subspace of a normed linear space X, then for bounded set C we put

$$\mathcal{S}_W(C) = \{w_0 \in W : d(C, W) = \sup_{c \in C} \|c - w_0\|\}$$

the set of all best simultaneous approximations for C from W. It is clear that $\mathcal{S}_W(C)$ is a bounded and convex subset of X and if W is a simultaneous proximinal subspace in X, then W is closed in X.

Proposition 2.1. *Let X be a reflexive space, C a bounded set in X. Then every closed subspace of X is a simultaneous proximinal subspace of X.*

Proof. Suppose W is a closed subspace of X. For each $n \in \mathbf{N}$, there exists a sequence $\{w_n\}$ such that

$$\sup_{c\in C} ||c - w_n|| \leq d(C, W) + \frac{1}{n}. \ (*)$$

Since X is a reflexive and $\{w_n\}$ is bounded there exist subsequence $\{w_{n_k}\}$ and w_0 such that $w_{n_k} \rightharpoonup w_0$. Therefore

$$\sup_{c\in C} ||c - w_0|| \leq d(C, W)$$

and but $d(C, W) \leq \sup_{c\in C} ||c - w_0||$, therefore

$$d(C, W) = \sup_{c\in C} ||c - w_0||.$$

Hence $w_0 \in \mathcal{S}_W(C)$. □

Theorem 2.2. *Let X be a reflexive Banach space, W a closed subspace of X and $1 < p < \infty$. Then $L^p(S, W)$ is a simultaneous proximinal subspace of $L^p(S, X)$.*

Proof. Since W is a closed subspace of X, $L^p(S, W)$ is a closed subspace of $L^p(S, X)$. On the other hand, since X is reflexive therefore by Lemma 1.1 $L^p(S, X)$ is reflexive. Then by Proposition 2.1, $L^p(S, W)$ is a simultaneous proximinal subspace of $L^p(S, X)$. □

Theorem 2.3. *Let W be a finite-dimensional subspace in a Banach space X. Then $L^\infty(S, W)$ is simultaneous proximinal in $L^\infty(S, X)$.*

Proof. Suppose $f \in L^\infty(S, X)$. For each $s \in S$ define

$$\Phi(s) = \{w_0 \in W \ : \ \sup_{f\in C} \|f(s) - w_0\| = \inf_{w\in W} \sup_{f\in C} \|f(s) - w\|\}.$$

For each s, $\Phi(s)$ is a closed, bounded, and nonempty subset of W. We shall show that for each compact subset K of W the set

$$K^* = \{s \in S : \Phi(s) \cap K \neq \emptyset\}$$

is measurable in S. The set K^* can also be described as

$$K^* = \{s \in S : \inf_{g\in K} \sup_{f\in C} \|f(s) - g\| = \inf_{g\in W} \sup_{f\in C} \|f(s) - g\|\}.$$

Since subtraction in X and the norm in X are continuous, the mapping $s \mapsto \|f(s) - g\|$ is measurable for each g. Hence the mapping

$$s \mapsto \inf_{g\in K} \sup_{f\in C} \|f(s) - g\|$$

is measurable for any set A. It follows that K^* is measurable. By Theorem 11.17 of [5] there is a measurable function $\phi : S \to W$ such that $\phi(s) \in \Phi(s)$ for each $s \in S$. Since W is finite-dimensional, it is separable. Hence by Lemma 10.3 of [5] ϕ is strongly measurable. Since $\|\phi(s)\| \leq 2\sup_{f\in C}\|f(s)\|$, it follows that $\|\phi\| \leq 2\sup_{f\in C}\|f\|$. Thus $\phi \in L^\infty(S,W)$. For any $k \in L^\infty(S,W)$ we have

$$\begin{aligned} \sup_{f\in C}\|f(s)-\phi(s)\| &= inf_{g\in K}\sup_{f\in C}\|f(s)-g\| \\ &\leq sup_{f\in C}\|f(s)-k(s)\| \end{aligned}$$

for all $s \in S$. Hence $sup_{f\in C}\|f-\phi\| \leq sup_{f\in C}\|f-k\|$. This proves that ϕ is a best simultaneous approximation for C in $L^\infty(S,W)$. □

Since every simultaneous proximinal is a proximinal subspace, hence by last theorem we consequence following corollary.

Proposition 2.4. *Let W be a closed subspace of reflexive Banach space X and $1 < p < \infty$. Then $L^p(S,W)$ is a proximinal subspace of $L^p(S,X)$.*

Now in the following we give a result in simultaneous Chebyshev.

Theorem 2.5. *Let X be a reflexive space, W a closed subspace of X. If $L^p(S,W)$ is a simultaneous Chebyshev subspace of $L^p(S,X)$ for $1 < p < \infty$, then W is a simultaneous Chebyshev subspace of X.*

Proof. The proof is trivial. □

Proposition 2.6. *Let C be a compact subset of $L^p(S,X)$, $g_0(s) \in W$ a best simultaneous approximation from $\{f(s) : f \in C\}$ for all $s \in S$ and $1 < p < \infty$, then $g_0 \in L^1(S,W)$ is a best simultaneous approximation from C.*

Proof. Suppose that $g_0(s) \in W$ is a best simultaneous approximation from $\{f(s) : f \in C\}$, then

$$\sup_{f\in C}\|f(s)-g_0(s)\| = \inf_{w\in W}\sup_{f\in C}\|f(s)-w\| \ (*).$$

Therefore

$$\begin{aligned} \sup_{f\in C}\|f-g_0\|_1 &\leq \int \|f(s)-g(s)\|d\mu(s) \\ &\leq \int \inf_{w\in W}\sup_{f\in C}\|f(s)-w\| \\ &\leq \int \inf_{g\in L^p(S,W)}\sup_{f\in C}\|f(s)-g(s)\| \\ &\leq \inf_{g\in L^p(S,W)}\int \sup_{f\in C}\|f(s)-g(s)\| \\ &= \inf_{g\in L^p(S,W)}\sup_{f\in C}\int \|f(s)-g(s)\|, \end{aligned}$$

where since C is compact we have last equality, and so $g_0 \in L^1(S,W)$ is a best simultaneous approximation from C. □

Proposition 2.7. *Every w^*-compact subset of dual space X is simultaneous proximinal subset of X.*

PROOF. Suppose C is an arbitrary bounded subset of dual space X and K is a w^*-compact subset of dual space X. Put

$$B(C,r) := \{g \in X \ : \ \sup_{f\in C} \|f-g\| \leq r\}.$$

Therefore for every $n \in N$ there exist a $g_n \in B(C, d(C,K) + \frac{1}{n}) \bigcap K$. Hence by compactness of K there exist a subsequence $\{g_{n_k}\}$ and g_0 such that $g_{n_k} \xrightarrow{w^*} g_0$ and so for every $f \in C$ we have $g_{n_k} - f \xrightarrow{w^*} g_0 - f$. Hence for every $f \in C$,

$$\|f - g_0\| \leq \liminf_{k\to\infty} \|f - g_{n_k}\|.$$

Then

$$\sup_{f\in C} \|f - g_0\| \leq d(C,W).$$

□

In the following, we give characterizations of best w-simultaneous approximinality in $L^p(S,X)$.

Definition 2.1. Let W be a subspace of X and C a bounded set in X. If for each compact set C there is at least one best simultaneous approximation in W, then W is called a *w-simultaneous proximinal subspace* of X.

If for each compact set C there is a unique best simultaneous approximation in W, then W is called a *w-simultaneous Chebyshev subspace* of X.

Theorem 2.8. *Let X be a uniformly convex space, W a subspace of X. Then W is a w-simultaneous Chebyshev subspace of X if and only if $L^p(S,W)$ is a w-simultaneous Chebyshev subspace of $L^p(S,X)$ for $1 < p < \infty$.*

PROOF. Suppose W is a w-simultaneous Chebyshev subspace of X. Then W is proximinal and so is closed. Hence by Proposition 2.1 W is a simultaneous proximinal subspace of X. If $g_1, g_2 \in \mathcal{S}_{L^p(S,X)}(C)$ for compact set C, we have

$$\sup_{f\in C} \|f - g_1\| = \sup_{f\in C} \|f - g_2\| = d(C, L^p(S,W)). \ (*)$$

Hence

$$\sup_{f\in C} \|f - (\frac{g_1 + g_2}{2})\| = d(C, L^p(S,W)).$$

Since C is compact, there exists $f_0 \in C$ such that

$$\|f_0 - (\frac{g_1 + g_2}{2})\| = d(C, L^p(S, W)). \ (**)$$

But by $(*)$ we have $\|f_0 - g_1\| \leq d(C, L^p(S, W))$ and $\|f_0 - g_2\| \leq d(C, L^p(S, W))$. On the other hand by Lemma 1.2, $L^p(S, X)$ is strictly convex and so

$$\|f_0 - (\frac{g_1 + g_2}{2})\| < d(C, L^p(S, W))$$

that is contradict with $(**)$ unless $g_1 = g_2$.

Conversely, it is a consequence of the last Theorem. □

Theorem 2.9. *Let W be a separable w^*-closed subspace of dual space X. Then $L^1(S, W)$ is a w-simultaneous proximinal subspace of $L^1(S, X)$.*

Proof. Suppose $f : S \to Y$ strongly measurable and for open set O in X, $f^{-1}(O) \in \mathcal{A}$. For each $s \in S$, define

$$\Phi(s) = \{w_0 \in W \ : \ \sup_{f \in C} \|f(s) - w_0\| = \inf_{w \in W} \sup_{f \in C} \|f(s) - w\|\}.$$

Then, for each $s \in S$, $\Phi(s)$ is a nonempty closed subset of W. Suppose that K be a w^*-compact set in W. Then

$$\Phi(s) = \{s \in S \ : \ \Phi(s) \cap K \ \textit{is nonempty}\}.$$

By Proposition 2.7 K is simultaneous proximinal in X, hence

$$\Phi^{-1}(K) = \{s \in S \ : \ \inf_{w \in K} \sup_{f \in C} \|f(s) - w\| = \inf_{w \in W} \sup_{f \in C} \|f(s) - w\|\}.$$

Since norm is continues and $f^{-1}(O) \in \mathcal{A}$ for each open set O, the mapping $s \to \|f(s) - w\|$ is measurable. Hence the mapping $s \to \inf_{w \in K} \sup_{f \in C} \|f(s) - w\|$ is measurable whenever K lies in W. Thus $\Phi^{-1}(K)$ is measurable. Therefore by Theorem 11.17 of [5] there is a selection $\phi : S \to W$ such that $\phi(s) \in \Phi(s)$ for each $s \in S$ and $\phi^{-1}(K) \in \mathcal{A}$ for w^*-compact set K in W.

Since W is separable, take a countable dense set $\{w_i\}$ in W. Each open set $O \subseteq W$ can be written as

$$O = \cup_{n,m=1}^{\infty} \{C_{nm} : C_{nm} \subset O\},$$

where $C_{nm} = \{w \in W \ : \ \|w - w_n\| \leq 1/m\}$. Each C_{nm} is a w^*-compact set in W and so $\phi^{-1}(C_{nm})$ is measurable. Hence $\phi^{-1}(O)$ is measurable for each O open in W. Since ϕ has a separable rang, ϕ is strongly measurable by Lemma 10.3 of [5]. Therefore by Theorem 2.8, ϕ is a best simultaneous approximation to C from $L^1(S, W)$. □

Corollary 2.10. *Let W be a finite-dimensional subspace of Banach space X. Then $L^1(S, W)$ is w-simultaneous proximinal.*

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