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Bifuzzy d-algebras under norms

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ABSTRACT. In this paper, by using norms (t-norms and t-conorms), we introduce the notions of bifuzzy d-algebras and bifuzzy d-ideals of d-algebras and investigate several interesting properties of them. Next, we consider the intersection and product of them. Finally we obtain some results about them under d-algebra homomorphisms.

1. Introduction

The notion of d-algebras was introduced and investigated by Neggers and Kim [**10**]. The concept of d-ideals in d-algebra was introduced by Neggers, et al. [**11**]. The concept of fuzzy sets was introduced by Zadeh [**31**]. The notion of fuzzy subalgebras and d-ideals in d-algebras was introduced by Akram and Dar [**2**]. Intuitionistic fuzzy set (in short IFS) introduced by Atanassov [**3**] as a generalization of fuzzy set theory enables us to describe this difference. IFS theory is conveniently and successfully applied in Abstract Algebra. The concept of intuitionistic fuzzy d-algebras was introduced and investigated by Jun, et al. [**8**]. The concept of intuitionistic fuzzy Dot d-ideals of d-algebras was introduced by Barbhuiya and Basnet [**4**]. The concept of intuitionistic fuzzy d-ideals of d-algebras was introduced by Hasan [**7**]. The norms, originated from the studies of probabilistic metric spaces. The author by using norms, investigated some properties of fuzzy algebraic structures [**10**]-[**30**]. In this paper, we define bifuzzy subalgebras of d-algebra X under norms (t-norm T and t-conorm C) as $BFSN(X)$ and bifuzzy d-ideals of d-algebra X under norms (t-norm T and t-conorm C) as $BFDIN(X)$ and characterize them. Next we prove that if $A, B \in BFSN(X)$ and $C, D \in BFDIN(X)$, then $A \cap B \in BFSN(X)$

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and $C \cap D \in BFDIN(X)$. Also we prove that $A \times B \in BFSN(X \times Y)$ and $C \times D \in BFDIN(X \times Y)$. Finally we prove that $\varphi(A) \in BFSN(Y)$ and $\varphi^{-1}(B) \in BFSN(X)$ also $\varphi(C) \in BFDIN(Y)$ and $\varphi^{-1}(D) \in BFSN(X)$ under d-algebra homomorphism $\varphi : X \to Y$.

2. Preliminaries

In this section we cite the fundamental definitions and results that will be used in the sequel:

Definition 2.1. (See [10]) A nonempty set X with a constant 0 and a binary operation $*$ is called a d-algebra, if it satisfies the following axioms:
(1) $x * x = 0$,
(2) $0 * x = 0$,
(3) if $x * y = 0$ and $y * x = 0$, then $x = y$,
for all $x, y \in X$.

Definition 2.2. (See [11]) Let S be a non-empty subset of a d-algebra X, then S is called subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Definition 2.3. (See [11]) Let X be a d-algebra and I be a subset of X, then I is called d-ideal of X if it satisfies following conditions:
(1) $0 \in I$,
(2) if $x * y \in I$ and $y \in I$, then $x \in I$,
(3) if $x \in I$ and $y \in X$, then $x * y \in I$.

Definition 2.4. (See [10]) A mapping $f : X \to Y$ of d-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$.

Definition 2.5. (See [9]) Let X be an arbitrary set. A fuzzy subset of X, we mean a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the $[0, 1]$-power set of X and is denoted $[0, 1]^X$. For a fixed $s \in [0, 1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ and the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is called a lower level of μ.

Definition 2.6. (See [3, 6]) Let X be a nonempty set. An intuitionistic fuzzy set is an object of the form: $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ such that $\mu_A, \nu_A \in [0, 1]^X$ and for all $x \in X$ we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. This object is also called a bifuzzy set. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$.
We denote the set of all bifuzzy sets of X under S by $BF(X)$.

Definition 2.7. (See [10]) Let $A = (\mu_A, \nu_A) \in BF(X)$ and $B = (\mu_B, \nu_B) \in BF(X)$. Define

$$A \cap B = (\mu_{A \cap B}, \nu_{A \cap B}) : X \to [0, 1]$$

as $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = C(\nu_A(x), \nu_B(x))$ for all $x \in X$.

Definition 2.8. (See [10]) Let $A = (\mu_A, \nu_A) \in BF(X)$ and $B = (\mu_B, \nu_B) \in BF(Y)$. The cartesian product of A and B is denoted by $A \times B : X \times Y \to [0,1]$ is defined by

$$(A \times B)(x,y) = ((\mu_A, \nu_A) \times (\mu_B, \nu_B))(x,y) = (\mu_{A\times B}, \nu_{A\times B})(x,y)$$
$$= (\mu_{A\times B}(x,y), \nu_{A\times B}(x,y)) = (T(\mu_A(x), \mu_B(y)), C(\nu_A(x), \nu_B(y)))$$

for all $(x,y) \in X \times Y$.

Definition 2.9. (See [3]) Let φ be a function from set X into set Y such that $A = (\mu_A, \nu_A) \in BF(X)$ and $B = (\mu_B, \nu_B) \in BF(Y)$. For all $x \in X, y \in Y$, with $\varphi^{-1}(y) \neq \emptyset$ we define $\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y))$ such that

$$\varphi(\mu_A)(y) = \sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\}$$

and

$$\varphi(\nu_A)(y) = \inf\{\mu_A(x) \mid x \in X, \varphi(x) = y\}.$$

And if $\varphi^{-1}(y) = \emptyset$, then $\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y)) = (0,1)$.
Also

$$\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x))).$$

Definition 2.10. (See [5]) A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x, T(y,z)) = T(T(x,y), z)$ (associativity),
for all $x, y, z \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.11. (1) Standard intersection t-norm $T_m(x,y) = \min\{x,y\}$.
(2) Bounded sum t-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product t-norm $T_p(x,y) = xy$.
(4) Drastic t-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y)=\begin{cases} 0 & \text{if } x=y=0 \\ \frac{xy}{x+y-xy} & \text{otherwise.}\end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y)\le T(x,y)\le T_{\min}(x,y)$ for all $x,y\in[0,1]$.

Definition 2.12. (See [5]) A t-norm C is a function $C:[0,1]\times[0,1]\to[0,1]$ having the following four properties:
(1) $C(x,0)=x$,
(2) $C(x,y)\le C(x,z)$ if $y\le z$,
(3) $C(x,y)=C(y,x)$,
(4) $C(x,C(y,z))=C(C(x,y),z)$,
for all $x,y,z\in[0,1]$.

We say that T and C be idempotent if for all $x\in[0,1]$ we have $T(x,x)=x$ and $C(x,x)=x$.

Example 2.13. The basic t-conorms are

$$C_m(x,y)=\max\{x,y\},$$
$$C_b(x,y)=\min\{1,x+y\}$$

and

$$C_p(x,y)=x+y-xy$$

for all $x,y\in[0,1]$.
S_m is standard union, C_b is bounded sum, C_p is algebraic sum.

Lemma 2.1. *(See [1]) Let C be a t-conorm and T be a t-norm. Then*

$$C(C(x,y),C(w,z))=C(C(x,w),C(y,z)),$$

and

$$T(T(x,y),T(w,z))=T(T(x,w),T(y,z)),$$

for all $x,y,w,z\in[0,1]$.

3. Bifuzzy d-algebras under norms

Throughout this section we let that X,Y will be two d-algebras.

Definition 3.1. Let $A=(\mu_A,\nu_A)\in BF(X)$ then A is called a bifuzzy subalgebra of X under norms (t-norm T and t-conorm C) if
(1) $\mu_A(x*y)\ge T(\mu_A(x),\mu_A(y))$,
(2) $\nu_A(x*y)\le C(\nu_A(x),\nu_A(y))$
for all $x,y\in X$.

Denote by $BFSN(X)$, the set of all bifuzzy subalgebras of X under norms (t-norm T and t-conorm C).

Example 3.2. Let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:

$*$	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Then $(X, *, 0)$ is a d-algebra.
Define fuzzy subset $\mu_A : (X, *, 0) \to [0, 1]$ as

$$\mu_A(x) = \begin{cases} 0.25 & \text{if } x = 0 \\ 0.55 & \text{if } x \neq 0 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.65 & \text{if } x = 0 \\ 0.15 & \text{if } x \neq 0. \end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ and $C(a, b) = C_p(a, b) = a + b - ab$, for all $a, b \in [0, 1]$, then $A = (\mu_A, \nu_A) \in BFSN(X)$.

Proposition 3.1. *Let $A = (\mu_A, \nu_A) \in BF(X)$ and T, C be idempotent. Then $A = (\mu_A, \nu_A) \in BFSN(X)$ if and only if the $A_{s,t} = \{x \in X : \mu_A(x) \geq s, \nu_A(x) \leq t\}$ is either empty or a subalgebra of X for every $t \in [0, 1]$.*

Proof. Let $A = (\mu_A, \nu_A) \in BFSN(X)$ and $x, y \in A_{s,t}$. Then

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) \geq T(s, s) = s$$

and

$$\nu_A(x * y) \leq C(\nu_A(x), \nu_A(y)) \leq C(t, t) = t$$

thus $x * y \in A_{s,t}$ and so $A_{s,t}$ will be a subalgebra of X for every $t \in [0, 1]$.
Conversely, let $A_{s,t}$ is either empty or a subalgebra of X for every $t \in [0, 1]$. Let $s = T(\mu_A(x), \mu_A(y))$ and $t = C(\nu_A(x), \nu_A(y))$ and $x, y \in A_{s,t}$. As $A_{s,t}$ is a subalgebra of X so $x * y \in A_{s,t}$ and thus

$$\mu_A(x * y) \geq s = T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(x * y) \leq t = C(\nu_A(x), \nu_A(y))$$

so $A = (\mu_A, \nu_A) \in BFSN(X)$. □

Proposition 3.2. *Let M be a subalgebra of a d-algebra X and $A = (\mu_A, \nu_A) \in BF(M)$ with*

$$A(x) = (\mu_A(x), \nu_A(x)) = \begin{cases} (s, t) & \textit{if } x \in M \\ (0, 1) & \textit{if } x \notin M \end{cases}$$

and $s, t \in (0, 1)$.
If T, C *be idempotent, then* $A = (\mu_A, \nu_A) \in BFSN(X)$.

PROOF. Undoubtedly $A = A_{s,t}$. Let $x, y \in X$ and we have the followin conditions.
(1) If $x, y \in M$, then $x * y \in M$ and so

$$\mu_A(x * y) = s \geq s = T(s, s) = T(\mu_A(x), \mu_A(x))$$

and

$$\nu_A(x * y) = t \leq t = C(t, t) = C(\nu_A(x), \nu_A(x)).$$

(2) If If $x \in M$ and $y \notin M$, then $\mu_A(x) = s$ and $\mu_A(y) = 0$ then

$$\mu_A(x * y) \geq 0 = T(s, 0) = T(\mu_A(x), \mu_A(y)).$$

Also $\nu_A(x) = t$ and $\nu_A(y) = 1$ then

$$\nu_A(x * y) \leq 1 = C(t, 1) = C(\nu_A(x), \nu_A(y)).$$

(3) If $x \notin M$ and $y \in M$ then it will be similar to (2).
(4) If $x \notin M$ and $y \notin M$, then $\mu_A(x) = 0$ and $\mu_A(y) = 0$ and so

$$\mu_A(x * y) \geq 0 = T(0, 0) = T(\mu_A(x), \mu_A(y)).$$

Also $\nu_A(x) = 1$ and $\nu_A(y) = 1$ and so

$$\nu_A(x * y) \leq 1 = C(1, 1) = C(\nu_A(x), \nu_A(y)).$$

Now as (1)-(4) we get that $A = (\mu_A, \nu_A) \in BFSN(X)$. □

Corollary 3.3. *Let* M *be a subset of* X. *Let*

$$\chi_M(x) = (\mu_A(x), \nu_A(x)) = \begin{cases} (1, 0) & \text{if } x \in M \\ (0, 1) & \text{if } x \notin M \end{cases}$$

be the characteristic function. Then $\chi_M \in BFSN(X)$ *if and only if* M *is a subalgebra of* X.

Proposition 3.4. *Let* $A = (\mu_A, \nu_A) \in BFSN(X)$ *and* $B = (\mu_B, \nu_B) \in BFSN(X)$. *then* $A \cap B \in BFSN(X)$.

PROOF. Let $x, y \in X$. Then

$$\begin{aligned} \mu_{A\cap B}(x * y) &= T(\mu_A(x * y), \mu_B(x * y)) \\ &\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))) \\ &= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y))) \\ &= T(\mu_{A\cap B}(x), \mu_{A\cap B}(y)). \end{aligned}$$

thus

$$\mu_{A\cap B}(x * y) \geq T(\mu_{A\cap B}(x), \mu_{A\cap B}(y)).$$

Also

$$\begin{aligned}\nu_{A\cap B}(x*y) &= C(\nu_A(x*y),\nu_B(x*y))\\ &\le C(C(\nu_A(x),\nu_A(y)),C(\nu_B(x),\nu_B(y)))\\ &= C(C(\nu_A(x),\nu_B(x)),C(\nu_A(y),\nu_B(y)))\\ &= C(\nu_{A\cap B}(x),\nu_{A\cap B}(y)).\end{aligned}$$

then

$$\nu_{A\cap B}(x*y)\le C(\nu_{A\cap B}(x),\nu_{A\cap B}(y)).$$

Thus $A\cap B=(\mu_{A\cap B},\nu_{A\cap B})\in BFSN(X)$. □

Proposition 3.5. *If $A=(\mu_A,\nu_A)\in BFSN(X)$ and $\varphi: X\to Y$ be a homomorphism of d-algebras, then $\varphi(A)\in BFSN(Y)$.*

Proof. Let $y_1,y_2\in Y$ and $x_1,x_2\in X$ such that $\varphi(x_1)=y_1$ and $\varphi(x_2)=y_2$. Then

$$\begin{aligned}\varphi(\mu_A)(y_1*y_2) &= \sup\{\mu_A(x_1*x_2)\mid x_1,x_2\in X,\varphi(x_1)=y_1,\varphi(x_2)=y_2\}\\ &\ge \sup\{T(\mu_A(x_1),\mu_A(x_2)\mid x_1,x_2\in X,\varphi(x_1)=y_1,\varphi(x_2)=y_2\}\\ &= T(\sup\{\mu_A(x_1)|x_1\in X,\varphi(x_1)=y_1\},\sup\{\mu_A(x_2)|x_2\in X,\varphi(x_2)=y_2\})\\ &= T(\varphi(\mu_A)(y_1),\varphi(\mu_A)(y_2)).\end{aligned}$$

Thus

$$\varphi(\mu_A)(y_1*y_2)\ge T(\varphi(\mu_A)(y_1),\varphi(\mu_A)(y_2)).$$

Also

$$\begin{aligned}\varphi(\nu_A)(y_1*y_2) &= \inf\{\nu_A(x_1*x_2)\mid x_1,x_2\in X,\varphi(x_1)=y_1,\varphi(x_2)=y_2\}\\ &\le \inf\{C(\nu_A(x_1),\nu_A(x_2)\mid x_1,x_2\in X,\varphi(x_1)=y_1,\varphi(x_2)=y_2\}\\ &= C(\inf\{\nu_A(x_1)|x_1\in X,\varphi(x_1)=y_1\},\inf\{\nu_A(x_2)|x_2\in X,\varphi(x_2)=y_2\})\\ &= C(\varphi(\nu_A)(y_1),\varphi(\nu_A)(y_2))\end{aligned}$$

so

$$\varphi(\nu_A)(y_1*y_2)\le C(\varphi(\nu_A)(y_1),\varphi(\nu_A)(y_2)).$$

Then $\varphi(A)=(\varphi(\mu_A),\varphi(\nu_A))\in BFSN(Y)$. □

Proposition 3.6. *If $B=(\mu_B,\nu_B)\in BFSN(Y)$ and $\varphi: X\to Y$ be a homomorphism of d-algebras, then $\varphi^{-1}(B)\in BFSN(X)$.*

Proof. Let $x_1,x_2\in X$. Then

$$\begin{aligned}\varphi^{-1}(\mu_B)(x_1*x_2) &= \mu_B(\varphi(x_1*x_2))\\ &= \mu_B(\varphi(x_1)*\varphi(x_2))\\ &\ge T(\mu_B(\varphi(x_1)),\mu_B(\varphi(x_2)))\\ &= T(\varphi^{-1}(\mu_B)(x_1),\varphi^{-1}(\mu_B)(x_2))\end{aligned}$$

thus

$$\varphi^{-1}(\mu_B)(x_1 * x_2) \geq T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2)).$$

Also

$$\begin{aligned}\varphi^{-1}(\nu_B)(x_1 * x_2) &= \nu_B(\varphi(x_1 * x_2)) \\ &= \nu_B(\varphi(x_1) * \varphi(x_2)) \\ &\leq C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) \\ &= C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))\end{aligned}$$

then

$$\varphi^{-1}(\nu_B)(x_1 * x_2) \leq C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2)).$$

Therefore $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in BFSN(X)$. □

Proposition 3.7. *Let $A = (\mu_A, \nu_A) \in BFSN(X)$ and $B = (\mu_B, \nu_B) \in BFSN(Y)$. Then $A \times B \in BFSN(X \times Y)$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}(\mu_{A\times B})((x_1, y_1) * (x_2, y_2)) &= (\mu_{A\times B})(x_1 * x_2, y_1 * y_2) \\ &= T(\mu_A(x_1 * x_2), \mu_B(y_1 * y_2)) \\ &\geq T(T(\mu_A(x_1), \mu_A(x_2)), T(\mu_B(y_1), \mu_B(y_2))) \\ &= T(T(\mu_A(x_1), \mu_B(y_1)), T(\mu_A(x_2), \mu_B(y_2))) \text{ (Lemma 2.1)} \\ &= T(\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2))\end{aligned}$$

thus

$$\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq T(\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)).$$

Also

$$\begin{aligned}(\nu_{A\times B})((x_1, y_1) * (x_2, y_2)) &= (\nu_{A\times B})(x_1 * x_2, y_1 * y_2) \\ &= C(\nu_A(x_1 * x_2), \nu_B(y_1 * y_2)) \\ &\leq C(C(\nu_A(x_1), \nu_A(x_2)), C(\nu_B(y_1), \nu_B(y_2))) \\ &= C(C(\nu_A(x_1), \nu_B(y_1)), C(\nu_A(x_2), \nu_B(y_2))) \text{ (Lemma 2.1)} \\ &= C(\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2))\end{aligned}$$

then

$$\nu_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq C(\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)).$$

Therefore

$$A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in BFSN(X \times Y).$$

□

4. Bifuzzy d-ideals of d-algebras under norms

Definition 4.1. We say that $A = (\mu_A, \nu_A) \in BF(X)$ is bifuzzy d-ideal of X under norms(t-norm T and t-conorm C) if it satisfies the following inequalities:
(1) $\mu_A(0) \geq \mu_A(x)$,
(2) $\mu_A(x) \geq T(\mu_A(x * y), \mu_A(y))$,
(3) $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$,
(4) $\nu_A(0) \leq \nu_A(x)$,
(5) $\nu_A(x) \leq C(\nu_A(x * y), \nu_A(y))$,
(6) $\nu_A(x * y) \leq C(\nu_A(x), \nu_A(y))$,
for all $x, y \in X$.
Denote by $BFDIN(X)$, the set of all bifuzzy d-ideals of X under norms(t-norm T and t-conorm C).

Example 4.2. Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then $(X, *, 0)$ is a d-algebra. Define $A = (\mu_A, \nu_A) \in BF(X)$ as

$$\mu_A(x) = \begin{cases} 0.35 & \text{if } x = 0 \\ 0.55 & \text{if } x \neq 0 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.45 & \text{if } x = 0 \\ 0.25 & \text{if } x \neq 0. \end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ and $C(a, b) = C_p(a, b) = a + b - ab$, for all $a, b \in [0, 1]$, then $A = (\mu_A, \nu_A) \in BFDIN(X)$

Proposition 4.1. *Let $A = (\mu_A, \nu_A) \in BFDIN(X)$ and $B = (\mu_B, \nu_B) \in BFDIN(Y)$. Then $A \times B \in BFDIN(X \times Y)$.*

Proof. (1) Let $(x, y) \in X \times Y$. Then

$$\mu_{A\times B}(0, 0) = T(\mu_A(0),)\mu_B(0)) \geq T(\mu_A(x), (\mu_B(y)) = \mu_{A\times B}(x, y).$$

Let $x_i \in X$ and $y_i \in Y$ for $i = 1, 2$.
(2)

$$\begin{aligned}\mu_{A\times B}(x_1, y_1) &= T(\mu_A(x_1), \mu_B(y_1))\\ &\geq T(T(\mu_A(x_1 * x_2), \mu_A(x_2)), T(\mu_B(y_1 * y_2), \mu_B(y_2)))\\ &= T(T(\mu_A(x_1 * x_2), \mu_B(y_1 * y_2)), T(\mu_A(x_2), \mu_B(y_2))) \textit{ (Lemma 2.1)}\\ &= T(\mu_{A\times B}(x_1 * x_2, y_1 * y_2), \mu_{A\times B}(x_2, y_2))\\ &= T(\mu_{A\times B}((x_1, y_1) * (x_2, y_2)), \mu_{A\times B}(x_2, y_2))\end{aligned}$$

then

$$\mu_{A\times B}(x_1, y_1) \geq T(\mu_{A\times B}((x_1, y_1) * (x_2, y_2)), \mu_{A\times B}(x_2, y_2)).$$

(3)

$$\begin{aligned}\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) &= \mu_{A\times B}(x_1 * x_2, y_1 * y_2)\\ &= T(\mu_A(x_1 * x_2), \mu_B(y_1 * y_2))\\ &\geq T(T(\mu_A(x_1), \mu_A(x_2)), T(\mu_B(y_1), \mu_B(y_2)))\\ &= T(T(\mu_A(x_1), \mu_B(y_1)), T(\mu_A(x_2), \mu_B(y_2))) \textit{ (Lemma 2.1)}\\ &= T(\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2))\end{aligned}$$

thus

$$\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq T(\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)).$$

(4) Let $(x, y) \in X \times Y$. Then

$$\nu_{A\times B}(0, 0) = C(\nu_A(0),)\nu_B(0)) \leq C(\nu_A(x), (\nu_B(y)) = \nu_{A\times B}(x, y).$$

Let $x_i \in X$ and $y_i \in Y$ for $i = 1, 2$.
(5)

$$\begin{aligned}\nu_{A\times B}(x_1, y_1) &= C(\nu_A(x_1), \nu_B(y_1))\\ &\leq C(C(\nu_A(x_1 * x_2), \nu_A(x_2)), C(\nu_B(y_1 * y_2), \nu_B(y_2)))\\ &= C(C(\nu_A(x_1 * x_2), \nu_B(y_1 * y_2)), C(\nu_A(x_2), \nu_B(y_2))) \textit{ (Lemma 2.1)}\\ &= C(\nu_{A\times B}(x_1 * x_2, y_1 * y_2), \nu_{A\times B}(x_2, y_2))\\ &= C(\nu_{A\times B}((x_1, y_1) * (x_2, y_2)), \nu_{A\times B}(x_2, y_2))\end{aligned}$$

then

$$\nu_{A\times B}(x_1, y_1) \leq C(\nu_{A\times B}((x_1, y_1) * (x_2, y_2)), \nu_{A\times B}(x_2, y_2)).$$

(6)

$$\begin{aligned}\nu_{A\times B}((x_1,y_1)*(x_2,y_2)) &= \nu_{A\times B}(x_1*x_2, y_1*y_2)\\ &= C(\nu_A(x_1*x_2), \nu_B(y_1*y_2))\\ &\leq C(C(\nu_A(x_1),\nu_A(x_2)), C(\nu_B(y_1),\nu_B(y_2)))\\ &= C(C(\nu_A(x_1),\nu_B(y_1)), C(\nu_A(x_2),\nu_B(y_2)))\ \textit{(Lemma 2.1)}\\ &= C(\nu_{A\times B}(x_1,y_1), \nu_{A\times B}(x_2,y_2))\end{aligned}$$

thus

$$\nu_{A\times B}((x_1,y_1)*(x_2,y_2)) \leq C(\nu_{A\times B}(x_1,y_1), \nu_{A\times B}(x_2,y_2)).$$

Thus as (1)-(6) we will have that $A\times B = (\mu_{A\times B}, \nu_{A\times B}) \in BFDIN(X\times Y)$. □

Proposition 4.2. *Let $A = (\mu_A, \nu_A) \in BF(X)$ and $B = (\mu_B, \nu_B) \in BF(Y)$. If $A\times B \in BFDIN(X\times Y)$, then at least one of the following two statements must hold.*
(1) $A(0) \supseteq A(x)$ for all $x \in X$.
(2) $B(0) \supseteq B(y)$ for all $y \in Y$.

Proof. Let the statements (1) and (2) are not holds, then we can find $(x, y) \in X\times Y$ such that $A(0) \subset A(x)$ and $B(0) \subset B(y)$. Then $\mu_A(0) < \mu_A(x)$ and $\nu_A(0) > \nu_A(x)$ and $\mu_B(0) < \mu_B(y)$ and $\nu_B(0) > \nu_B(y)$. Now

$$\mu_{A\times B}(x,y) = T(\mu_A(x), \mu_B(y)) > T(\mu_A(0), \mu_B(0)) = \mu_{A\times B}(0,0)$$

and

$$\nu_{A\times B}(x,y) = C(\nu_A(x), \nu_B(y)) < C(\nu_A(0), \nu_B(0)) = \nu_{A\times B}(0,0)$$

give us that we have contradiction with $A\times B \in BFDIN(X\times Y)$. □

Proposition 4.3. *Let $A = (\mu_A, \nu_A) \in BF(X)$ and $B = (\mu_B, \nu_B) \in BF(Y)$. If $A\times B \in BFDIN(X\times Y)$, and T, C be idempotent, then we obtain the following statements:*
(1) If $A(0) \supseteq A(x)$, then either $B(0) \supseteq A(x)$ or $B(0) \supseteq B(y)$ for all $(x, y) \in X\times Y$.
(2) If $B(0) \supseteq B(y)$, then either $A(0) \supseteq B(y)$ or $A(0) \geq A(x)$ for all $(x, y) \in X\times Y$.

Proof. (1) Let $A(0) \supseteq A(x)$ then $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$. Let $(x, y) \in X\times Y$ such that $B(0) \subset A(x)$ and $B(0) \subset B(y)$ then $\mu_B(0) < \mu_A(x)$ and $\nu_B(0) > \nu_A(x)$ and $\mu_B(0) < \mu_B(y)$ and $\nu_B(0) > \nu_A(y)$. Thus $\mu_A(0) \geq \mu_A(x) > \mu_B(0)$ and $\nu_A(0) \leq \nu_A(x) < \nu_B(0)$ which mean that $\mu_B(0) = T(\mu_A(0), \mu_B(0))$ and $\nu_B(0) = C(\nu_A(0), \nu_B(0))$. Now

$$\begin{aligned}\mu_{A\times B}(x,y) &= T(\mu_A(x), \mu_B(y))\\ &> T(\mu_B(0), \mu_B(0)) = \mu_B(0)\\ &= T(\mu_A(0), \mu_B(0)) = \mu_{A\times B}(0,0)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\times B}(x,y) &= C(\nu_A(x),\nu_B(y))\\ &< C(\nu_B(0),\nu_B(0)) = \nu_B(0)\\ &= C(\nu_A(0),\nu_B(0)) = \nu_{A\times B}(0,0)\end{aligned}$$

thus we will have contradiction with $A\times B \in BFDIN(X\times Y)$.
(2) It is similar to (1). □

Proposition 4.4. *Let $A=(\mu_A,\nu_A)\in BF(X)$ and $B=(\mu_B,\nu_B)\in BF(Y)$. If $A\times B\in BFDIN(X\times Y)$ and T,C be idempotent, then $A\in BFDIN(X)$ or $B\in BFDIN(Y)$.*

Proof. It is enough that $A\in BFDIN(X)$ and the proof $B\in BFDIN(Y)$ is similar. As Proposition 4.3 we have that $A(0)\supseteq A(x)$ for all $x\in X$ and from Proposition 4.2 part(1) we have either either $B(0)\supseteq A(x)$ or $B(0)\supseteq B(y)$ for all $(x,y)\in X\times Y$. Then we will have that

$$\mu_A(0)\geq \mu_A(x) \text{ (1)}$$

and

$$\nu_A(0)\leq \nu_A(x) \text{ (2)}$$

and $\mu_B(0)\geq\mu_A(x)$ and $\nu_B(0)\leq\nu_A(x)$ and $\mu_B(0)\geq\mu_B(y)$ and $\nu_B(0)\leq\nu_B(y)$ for all $(x,y)\in X\times Y$. Thus $\mu_{A\times B}(x,0)=T(\mu_A(x),\mu_B(0))=\mu_A(x)$ and $\nu_{A\times B}(x,0)=C(\nu_A(x),\nu_B(0))=\nu_A(x)$. Let $(x,y),(\acute{x},\acute{y})\in X\times Y$. Then and

$$\begin{aligned}\mu_{A\times B}(x,y) &\geq T(\mu_{A\times B}((x,y)*(\acute{x},\acute{y})),\mu_{A\times B}(\acute{x},\acute{y}))\\ &= T(\mu_{A\times B}((x*\acute{x},y*\acute{y})),\mu_{A\times B}(\acute{x},\acute{y}))\end{aligned}$$

and by putting $y=\acute{y}=0$ we get that

$$\mu_{A\times B}(x,0)\geq T(\mu_{A\times B}(x*\acute{x},0*0),\mu_{A\times B}(\acute{x},0))$$

and so

$$\mu_A(x)\geq T(\mu_A(x*\acute{x}),\mu_A(\acute{x})). \qquad (3)$$

Also

$$\begin{aligned}\nu_{A\times B}(x,y) &\leq C(\nu_{A\times B}((x,y)*(\acute{x},\acute{y})),\nu_{A\times B}(\acute{x},\acute{y}))\\ &= C(\nu_{A\times B}((x*\acute{x},y*\acute{y})),\nu_{A\times B}(\acute{x},\acute{y}))\end{aligned}$$

and as putting $y=\acute{y}=0$ we get that

$$\nu_{A\times B}(x,0)\leq C(\nu_{A\times B}(x*\acute{x},0*0),\nu_{A\times B}(\acute{x},0))$$

and so

$$\nu_A(x)\leq C(\nu_A(x*\acute{x}),\nu_A(\acute{x})). \qquad (4)$$

Since $A \times B \in BFDIN(X \times Y)$ so

$$\mu_{A\times B}((x,y)*(\acute{x},\acute{y})) \geq T(\mu_{A\times B}(x,y),\mu_{A\times B}(\acute{x},\acute{y}))$$

thus

$$\mu_{A\times B}(x*\acute{x},y*\acute{y}) \geq T(\mu_{A\times B}(x,y),\mu_{A\times B}(\acute{x},\acute{y}))$$

and by letting $y=\acute{y}=0$ we give that

$$\mu_{A\times B}(x*\acute{x},0*0) \geq T(\mu_{A\times B}(x,0),\mu_{A\times B}(\acute{x},0))$$

which means that

$$\mu_A(x*\acute{x}) \geq T(\mu_A(x),\mu_A(\acute{x})). \qquad (5)$$

Also

$$\nu_{A\times B}((x,y)*(\acute{x},\acute{y})) \leq C(\nu_{A\times B}(x,y),\nu_{A\times B}(\acute{x},\acute{y}))$$

thus

$$\nu_{A\times B}(x*\acute{x},y*\acute{y}) \leq C(\nu_{A\times B}(x,y),\nu_{A\times B}(\acute{x},\acute{y}))$$

and by letting $y=\acute{y}=0$ we give that

$$\nu_{A\times B}(x*\acute{x},0*0) \leq C(\nu_{A\times B}(x,0),\nu_{A\times B}(\acute{x},0))$$

which means that

$$\nu_A(x*\acute{x}) \leq C(\nu_A(x),\nu_A(\acute{x})). \qquad (6)$$

Therefore as (1)-(6) we get that $A=(\mu_A,\nu_A) \in BFDIN(X)$. □

Definition 4.3. Let $A=(\mu_A,\nu_A) \in BF(S)$ and $B=(\mu_B,\nu_B) \in BF(S)$, the strongest bifuzzy relation on S under norms(T,C) is bifuzzy relation on A as

$$B_A=(\mu_{B_A},\nu_{B_A}) : S\times S \to [0,1]\times[0,1]$$

as

$$\mu_{B_A}(x,y)=T(\mu_A(x),\mu_A(y))$$

and

$$\nu_{B_A}(x,y)=C(\nu_A(x),\nu_A(y))$$

for all $x,y \in S$.

Proposition 4.5. *Let T,C be idempotent. Then*

$$A=(\mu_A,\nu_A) \in BFDIN(X) \Longleftrightarrow B_A=(\mu_{B_A},\nu_{B_A}) \in BFDIN(X\times X).$$

Proof. Let $A=(\mu_A,\nu_A) \in BFDIN(X)$.
(1) Let $x \in X$ then

$$\mu_{B_A}(0,0)=T(\mu_A(0),\mu_A(0)) \geq T(\mu_A(x),\mu_A(x))=\mu_{B_A}(x,x)$$

and

$$\nu_{B_A}(0,0)=C(\nu_A(0),\nu_A(0)) \leq C(\nu_A(x),\nu_A(x))=\nu_{B_A}(x,x).$$

(2) Let $(x_1,x_2),(y_1,y_2) \in X\times X$. Then

$$\begin{aligned}\mu_{B_A}(x_1,x_2) &= T(\mu_A(x_1),\mu_A(x_2))\\ &\geq T(T(\mu_A(x_1*y_1),\mu_A(y_1)),T(\mu_A(x_2*y_2),\mu_A(y_2)))\\ &= T(T(\mu_A(x_1*y_1),\mu_A(x_2*y_2)),T(\mu_A(y_1),\mu_A(y_2)))\ \textit{(Lemma 2.1)}\\ &= T(\mu_{B_A}(x_1*y_1,x_2*y_2),\mu_{B_A}(y_1,y_2))\\ &= T(\mu_{B_A}((x_1,x_2)*(y_1,y_2)),\mu_{B_A}(y_1,y_2))\end{aligned}$$

thus

$$\mu_{B_A}(x_1,x_2) \geq T(\mu_{B_A}((x_1,x_2)*(y_1,y_2)),\mu_{B_A}(y_1,y_2))$$

and

$$\begin{aligned}\nu_{B_A}(x_1,x_2) &= C(\nu_A(x_1),\nu_A(x_2))\\ &\leq C(C(\nu_A(x_1*y_1),\nu_A(y_1)),C(\nu_A(x_2*y_2),\nu_A(y_2)))\\ &= C(C(\nu_A(x_1*y_1),\nu_A(x_2*y_2)),C(\nu_A(y_1),\nu_A(y_2)))\ \textit{(Lemma 2.1)}\\ &= C(\nu_{B_A}(x_1*y_1,x_2*y_2),\nu_{B_A}(y_1,y_2))\\ &= C(\nu_{B_A}((x_1,x_2)*(y_1,y_2)),\nu_{B_A}(y_1,y_2))\end{aligned}$$

then

$$\nu_{B_A}(x_1,x_2) \leq C(\nu_{B_A}((x_1,x_2)*(y_1,y_2)),\nu_{B_A}(y_1,y_2)).$$

(3) Let $(x_1,x_2),(y_1,y_2) \in X \times X$. Then

$$\begin{aligned}\mu_{B_A}((x_1,x_2)*(y_1,y_2)) &= \mu_{B_A}(x_1*y_1,x_2*y_2)\\ &= T(\mu_A(x_1*y_1),\mu_A(x_2*y_2))\\ &\geq T(T(\mu_A(x_1),\mu_A(y_1)),T(\mu_A(x_2),\mu_A(y_2)))\\ &= T(T(\mu_A(x_1),\mu_A(x_2)),T(\mu_A(y_1),\mu_A(y_2)))\ \textit{(Lemma 2.1)}\\ &= T(\mu_{B_A}(x_1,x_2),\mu_{B_A}(y_1,y_2))\end{aligned}$$

so

$$\mu_{B_A}((x_1,x_2)*(y_1,y_2)) \geq T(\mu_{B_A}(x_1,x_2),\mu_{B_A}(y_1,y_2))$$

and so

$$\nu_{B_A}((x_1,x_2)*(y_1,y_2)) \leq C(\nu_{B_A}(x_1,x_2),\nu_{B_A}(y_1,y_2)).$$

Therefore from (1)-(3) we get that $B_A = (\mu_{B_A},\nu_{B_A}) \in BFDIN(X \times X)$.
Conversely, let $B_A = (\mu_{B_A},\nu_{B_A}) \in BFDIN(X \times X)$.
(1) Let $x \in X$ then

$$\mu_A(0) = T(\mu_A(0),\mu_A(0)) = \mu_{B_A}(0,0) \geq \mu_{B_A}(x,x) = T(\mu_A(x),\mu_A(x)) = \mu_A(x)$$

and

$$\nu_A(0) = C(\nu_A(0),\nu_A(0)) = \nu_{B_A}(0,0) \leq \nu_{B_A}(x,x) = C(\nu_A(x),\nu_A(x)) = \nu_A(x).$$

Thus we get that

$$\mu_{B_A}(x,0) = T(\mu_A(x),\mu_A(0)) = \mu_A(x)$$

and

$$\nu_{B_A}(x,0) = C(\nu_A(x),\nu_A(0)) = \nu_A(x).$$

(2) Let $(x_1,x_2),(y_1,y_2) \in X \times X$. Thus

$$\begin{aligned}\mu_{B_A}(x_1,x_2) &\geq T(\mu_{B_A}((x_1,x_2)*(y_1,y_2)),\mu_{B_A}(y_1,y_2))\\ &= T(\mu_{B_A}(x_1*y_1,x_2*y_2),\mu_{B_A}(y_1,y_2))\end{aligned}$$

if we let $x_2 = y_2 = 0$ so

$$\mu_{B_A}(x_1,0) \geq T(\mu_{B_A}(x_1*y_1,0*0),\mu_{B_A}(y_1,0))$$

thus

$$\mu_A(x_1) \geq T(\mu_A(x_1*y_1),\mu_A(y_1))$$

and

$$\begin{aligned}\nu_{B_A}(x_1,x_2) &\leq C(\nu_{B_A}((x_1,x_2)*(y_1,y_2)),\nu_{B_A}(y_1,y_2))\\ &= C(\nu_{B_A}(x_1*y_1,x_2*y_2),\nu_{B_A}(y_1,y_2))\end{aligned}$$

if we let $x_2 = y_2 = 0$ so

$$\nu_{B_A}(x_1,0) \leq C(\nu_{B_A}(x_1*y_1,0*0),\nu_{B_A}(y_1,0))$$

then

$$\nu_A(x_1) \leq C(\nu_A(x_1*y_1),\nu_A(y_1)).$$

(3) Let $(x_1,x_2),(y_1,y_2) \in X \times X$. Now

$$\mu_{B_A}((x_1,x_2)*(y_1,y_2)) \geq T(\mu_{B_A}(x_1,x_2),\mu_{B_A}(y_1,y_2))$$

then

$$\mu_{B_A}(x_1*y_1,x_2*y_2) \geq T(\mu_{B_A}(x_1,x_2),\mu_{B_A}(y_1,y_2))$$

and by letting $x_2 = y_2 = 0$ we get that

$$\mu_{B_A}(x_1*y_1,0*0) \geq T(\mu_{B_A}(x_1,0),\mu_{B_A}(y_1,0))$$

and thus

$$\mu_A(x_1*y_1) \geq T(\mu_A(x_1),\mu_A(y_1))$$

and

$$\nu_{B_A}((x_1,x_2)*(y_1,y_2)) \leq C(\nu_{B_A}(x_1,x_2),\nu_{B_A}(y_1,y_2))$$

then

$$\nu_{B_A}(x_1*y_1,x_2*y_2) \leq C(\nu_{B_A}(x_1,x_2),\nu_{B_A}(y_1,y_2))$$

and by letting $x_2 = y_2 = 0$ we have that

$$\nu_{B_A}(x_1*y_1,0*0) \leq C(\nu_{B_A}(x_1,0),\nu_{B_A}(y_1,0))$$

and so

$$\nu_A(x_1*y_1) \leq C(\nu_A(x_1),\nu_A(y_1)).$$

Thus (1)-(3) give us that $A = (\mu_A, \nu_A) \in BFDIN(X)$. □

Proposition 4.6. *If* $A = (\mu_A, \nu_A) \in BFDIN(X)$ *and* $B = (\mu_B, \nu_B) \in BFDIN(X)$. *Then* $A \cap B \in BFDIN(X)$.

PROOF. Let $x, y \in X$. Then

(1)

$$\mu_{A\cap B}(0) = T(\mu_A(0), \mu_B(0)) \geq T(\mu_A(x), \mu_B(x)) = \mu_{A\cap B}(x).$$

(2)

$$\nu_{A\cap B}(0) = C(\nu_A(0), \nu_B(0)) \leq C(\nu_A(x), \nu_B(x)) = \nu_{A\cap B}(x).$$

(3)

$$\begin{aligned}\mu_{A\cap B}(x) &= T(\mu_A(x), \mu_B(x)) \\ &\geq T(T(\mu_A(x*y), \mu_A(y)), T(\mu_B(x*y), \mu_B(y))) \\ &= T(T(\mu_A(x*y), \mu_B(x*y)), T(\mu_A(y), \mu_B(y))) \\ &= T(\mu_{A\cap B}(x*y), \mu_{A\cap B}(y))\end{aligned}$$

and thus

$$\mu_{A\cap B}(x) \geq T(\mu_{A\cap B}(x*y), \mu_{A\cap B}(y)).$$

(4)

$$\begin{aligned}\nu_{A\cap B}(x) &= C(\nu_A(x), \nu_B(x)) \\ &\leq C(C(\nu_A(x*y), \nu_A(y)), C(\nu_B(x*y), \nu_B(y))) \\ &= C(C(\nu_A(x*y), \nu_B(x*y)), C(\nu_A(y), \nu_B(y))) \\ &= C(\nu_{A\cap B}(x*y), \nu_{A\cap B}(y))\end{aligned}$$

and so

$$\nu_{A\cap B}(x) \leq C(\nu_{A\cap B}(x*y), \nu_{A\cap B}(y)).$$

(5)

$$\begin{aligned}\mu_{A\cap B}(x*y) &= T(\mu_A(x*y), \mu_B(x*y)) \\ &\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))) \\ &= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y))) \\ &= T(\mu_{A\cap B}(x), \mu_{A\cap B}(y))\end{aligned}$$

thus

$$\mu_{A\cap B}(x*y) \geq T(\mu_{A\cap B}(x), \mu_{A\cap B}(y)).$$

(6)

$$\begin{aligned}\nu_{A\cap B}(x*y) &= C(\nu_A(x*y),\nu_B(x*y))\\ &\leq C(C(\nu_A(x),\nu_A(y)),C(\nu_B(x),\nu_B(y)))\\ &= C(C(\nu_A(x),\nu_B(x)),C(\nu_A(y),\nu_B(y)))\\ &= C(\nu_{A\cap B}(x),\nu_{A\cap B}(y))\end{aligned}$$

then

$$\nu_{A\cap B}(x*y)\leq C(\nu_{A\cap B}(x),\nu_{A\cap B}(y)).$$

Thus $A\cap B=(\mu_{A\cap B},\nu_{A\cap B})\in BFDIN(X)$. □

Proposition 4.7. *If $A=(\mu_A,\nu_A)\in BFDIN(X)$ and $\varphi: X\to Y$ be a homomorphism of d-algebras, then $\varphi(A)\in BFDIN(Y)$.*

Proof. (1) Let $x\in X$ and $y\in Y$ with $\varphi(x)=y$. Now

$$\begin{aligned}\varphi(\mu_A)(0) &= \sup\{\mu_A(0)\mid 0\in X,\varphi(0)=0\}\\ &\geq \sup\{\mu_A(x)\mid x\in X,\varphi(x)=y\}=\varphi(\mu_A)(y)\end{aligned}$$

and

$$\begin{aligned}\varphi(\nu_A)(0) &= \inf\{\nu_A(0)\mid 0\in X,\varphi(0)=0\}\\ &\leq \inf\{\nu_A(x)\mid x\in X,\varphi(x)=y\}=\varphi(\nu_A)(y).\end{aligned}$$

(2) Let $x,x_1\in X$ such that $f(x)=y,f(x_1)=y_1$. Then

$$\begin{aligned}\varphi(\mu_A)(y) &= \sup\{\mu_A(x)\mid x\in X,\varphi(x)=y\}\\ &\geq \sup\{T(\mu_A(x*x_1),\mu_A(x_1))\mid x,x_1\in X,\varphi(x)=y,\varphi(x_1)=y_1\}\\ &= T(\sup\{\mu_A(x*x_1)\mid x,x_1\in X,\varphi(x)=y,\varphi(x_1)=y_1\}\\ &\quad ,\sup\{\mu_A(x_1)\mid x_1\in X,\varphi(x_1)=y_1\})\\ &= T(\varphi(\mu_A)(y*y_1),\varphi(\mu_A)(y_1))\end{aligned}$$

therefore

$$\varphi(\mu_A)(y)\geq T(\varphi(\mu_A)(y*y_1),\varphi(\mu_A)(y_1)).$$

Also

$$\begin{aligned}\varphi(\nu_A)(y) &= \inf\{\nu_A(x)\mid x\in X,\varphi(x)=y\}\\ &\leq \inf\{C(\nu_A(x*x_1),\nu_A(x_1))\mid x,x_1\in X,\varphi(x)=y,\varphi(x_1)=y_1\}\\ &= C(\inf\{\nu_A(x*x_1)\mid x,x_1\in X,\varphi(x)=y,\varphi(x_1)=y_1\}\\ &\quad ,\inf\{\nu_A(x_1)\mid x_1\in X,\varphi(x_1)=y_1\})\\ &= C(\varphi(\nu_A)(y*y_1),\varphi(\nu_A)(y_1))\end{aligned}$$

thus

$$\varphi(\nu_A)(y)\leq C(\varphi(\nu_A)(y*y_1),\varphi(\nu_A)(y_1)).$$

(3) Let $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Then

$$\begin{aligned}\varphi(\mu_A)(y_1 * y_2) &= \sup\{\mu_A(x_1 * x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\ &\geq \sup\{T(\mu_A(x_1), \mu_A(x_2)) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\ &= T(\sup\{\mu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\} \\ &, \sup\{\mu_A(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\}) \\ &= T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2))\end{aligned}$$

thus

$$\varphi(\mu_A)(y_1 * y_2) \geq T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2)).$$

$$\begin{aligned}\varphi(\nu_A)(y_1 * y_2) &= \inf\{\nu_A(x_1 * x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\ &\leq \inf\{C(\nu_A(x_1), \nu_A(x_2)) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\ &= C(\inf\{\nu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\} \\ &, \inf\{\nu_A(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\}) \\ &= C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2))\end{aligned}$$

so

$$\varphi(\nu_A)(y_1 * y_2) \leq C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2)).$$

Then as (1)-(3), we get that $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in BFDIN(Y)$. □

Proposition 4.8. *If $B = (\mu_B, \nu_B) \in BFDIN(Y)$ and $\varphi : X \to Y$ be a homomorphism of d-algebras, then $\varphi^{-1}(B) \in BFDIN(X)$.*

Proof. (1) Let $x \in X$. Then

$$\varphi^{-1}(\mu_B)(0) = \mu_B(\varphi(0)) \geq \mu_B(\varphi(x)) = \varphi^{-1}(\mu_B)(x)$$

and

$$\varphi^{-1}(\nu_B)(0) = \nu_B(\varphi(0)) \leq \nu_B(\varphi(x)) = \varphi^{-1}(\nu_B)(x).$$

(2) Let $x, x_1 \in X$. As

$$\begin{aligned}\varphi^{-1}(\mu_B)(x) &= \mu_B(\varphi(x)) \\ &\geq T(\mu_B(\varphi(x) * \varphi(x_1)), \mu_B(\varphi(x_1))) \\ &= T(\mu_B(\varphi(x * x_1)), \mu_B(\varphi(x_1))) \\ &= T(\varphi^{-1}(\mu_B)(x * x_1), \varphi^{-1}(\mu_B)(x_1))\end{aligned}$$

so

$$\varphi^{-1}(\mu_B)(x) \geq T(\varphi^{-1}(\mu_B)(x * x_1), f^{-1}(\mu_B)(x_1))$$

and

$$\begin{aligned}\varphi^{-1}(\nu_B)(x) &= \nu_B(\varphi(x)) \\ &\leq C(\nu_B(\varphi(x)*\varphi(x_1)), \nu_B(\varphi(x_1))) \\ &= C(\nu_B(\varphi(x*x_1)), \nu_B(\varphi(x_1))) \\ &= C(\varphi^{-1}(\nu_B)(x*x_1), \varphi^{-1}(\nu_B)(x_1))\end{aligned}$$

thus

$$\varphi^{-1}(\nu_B)(x) \leq C(\varphi^{-1}(\nu_B)(x*x_1), \varphi^{-1}(\nu_B)(x_1)).$$

(3)

$$\begin{aligned}\varphi^{-1}(\mu_B)(x_1*x_2) &= \mu_B(\varphi(x_1*x_2)) \\ &= \mu_B(\varphi(x_1)*\varphi(x_2)) \\ &\geq T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2))) \\ &= T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))\end{aligned}$$

thus

$$\varphi^{-1}(\mu_B)(x_1*x_2) \geq T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2)).$$

Also

$$\begin{aligned}\varphi^{-1}(\nu_B)(x_1*x_2) &= \nu_B(\varphi(x_1*x_2)) \\ &= \nu_B(\varphi(x_1)*\varphi(x_2)) \\ &\leq C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) \\ &= C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))\end{aligned}$$

then

$$\varphi^{-1}(\nu_B)(x_1*x_2) \leq C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2)).$$

Therefore (1)-(3) give us that $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in BFDIN(X)$. □

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