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On various types of compatible Jungck–Rhoades pairs of mappings in C^*-algebra valued metric spaces

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Abstract. In this paper, among other things, we have established four different types of compatible mappings that work in the context of C^*-algebra valued metric spaces. The obtained types of mappings generalize from previously known ones within ordinary metric spaces. We have shown by examples that these types of mappings are really different. They can be used to consider new fixed point results which were done in the paper for the case of common fixed points of some mappings. The results in this paper generalize, extend, unify, enrich and complement many known results in the existing literature.

1. Introduction and preliminaries

Fixed point theory is the most interesting tool for various branches of non linear analysis. There are three main approaches in this theory: the metric, the topological and the order-theoretic approach, where representative examples are Banach's, Brouwer's and Tarski's theorems, respectively.
Fixed-point theory and its applications, such as numerical fixed-point theory and graphical fixed-point theory, will be truly groundbreaking in the coming years. At the intuitive level, fixed point theory can be used as an authoritative simulation method in different fields of science and/or technical sciences to derive solutions and/or experimental findings. From a more general point of view, fixed point theory can also be seen as an attempt to relate biological sciences/computational sciences to research into various abstract spaces on convergence analysis and compactness. Computer programming scientists research logic programming semantics using metric spaces (and/or their generalizations) because it is easy to formulate and can be

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defined and used to prove results. The current success of the establishment of various standardized metric spaces (and/or its associated results) has given rise to considerable interest and analysis in fixed-point metric theory (see e.g., [1],[2],[16],[17],[27]). On the other hand, the term of C^*-algebra was introduced by Segal [28] in 1947 to describe norm-closed sub algebras of $B(H)$, namely, the space of bounded operators on some Hilbert space H, $'C'$ stands for 'closed'. In [28], Segal considered a C^*-algebra as a "uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space". A real or a complex linear space $\mathcal{A}$ is said to be an algebra if vector multiplication is given for every pair of elements of $\mathcal{A}$ verifying two conditions so that $\mathcal{A}$ is a ring with respect to vector addition and vector multiplication, and for every scalar α and all $\Omega, \tau \in \mathcal{A}$, we have $\alpha(\Omega\tau) = (\alpha\Omega)\tau = \Omega(\alpha\tau)$. A norm $\|.\|$ on $\mathcal{A}$ is called sub-multiplicative if $\|\mu\varsigma\| \leq \|\mu\|\|\varsigma\|$ for all $\mu, \varsigma \in \mathcal{A}$. Here, $(\mathcal{A}, \|.\|)$ is said to be a normed algebra. A complete normed algebra is said to be a Banach algebra.

A $*$-algebra is a complex algebra with a linear involution $*$ so that $\tau^{**} = \tau$ and $(\tau z)^* = \tau^* z^*$, for all $\tau, z \in \mathcal{A}$. If a $*$-algebra is endowed with a complete sub-multiplicative norm verifying $\|\tau^*\| = \|\tau\|$ for each $\tau \in \mathcal{A}$, then the $*$-algebra is called a Banach $*$-algebra. A C^*-algebra is a Banach $*$-algebra so that $\|\tau^*\tau\| = \|\tau\|^2$ for each $\tau \in \mathcal{A}$. If a normed algebra $\mathcal{A}$ admits a unit $1_{\mathcal{A}}$ (that is, $\mu 1_{\mathcal{A}} = 1_{\mathcal{A}}\mu = \mu$ for each $\mu \in \mathcal{A}$ and $\|1_{\mathcal{A}}\| = 1$), then $\mathcal{A}$ is an unital normed algebra. A complete unital normed algebra $\mathcal{A}$ is said to be an unital Banach algebra. A positive element of $\mathcal{A}$ is an element $\sigma \in \mathcal{A}$ so that $\mu^* = \mu$ and its spectrum $\sigma(\mu) \subset \mathbb{R}_+$, where $\sigma(\mu) = \{\lambda \in \mathbb{R} : \lambda 1_{\mathcal{A}} - \mu \text{ is non-invertible}\}$. The set of all positive elements is denoted by $\mathcal{A}_+$. Such elements allow us to define a partial ordering '$\succeq$' on the elements of $\mathcal{A}$. That is,

$$\varsigma \succeq \mu \text{ if and only if } \varsigma - \mu \in \mathcal{A}_+.$$

If $\mu \in \mathcal{A}$ is positive, we write $\mu \succeq 0_{\mathcal{A}}$, where $0_{\mathcal{A}}$ is the zero element of $\mathcal{A}$. Each positive element μ of a C^*-algebra $\mathcal{A}$ has a unique positive square root. From now on, by $\mathcal{A}$ we mean an unital C^*-algebra with identity element $1_{\mathcal{A}}$. The sum of two positive elements in a C^*-algebra is a positive element. If μ is an arbitrary element of a C^*-algebra $\mathcal{A}$, then $\mu^*\mu$ is positive. Let $\mathcal{A}$ be a C^*-algebra and if $\mu, \varsigma \in \mathcal{A}^+$ such that $\mu \preceq \varsigma$, then for any $\Omega \in \mathcal{A}$, both $\Omega^*\mu\Omega$ and $\Omega^*\varsigma\Omega$ are positive and elements of $\mathcal{A}^+$ and $\Omega^*\mu\Omega \preceq \Omega^*\varsigma\Omega$. Further, $\mathbb{A}_+ = \{\mu \in \mathcal{A} : \mu \succeq 0_{\mathcal{A}}\}$ and $(\mu^*\mu)^{\frac{1}{2}} = |\mu|$.
A C^*-algebra $\mathcal{A}$ verifies the following algebraic operations:

1: addition, which is commutative and associative;
2: multiplication, which is associative;
3: multiplication by complex scalars;
4: an involution $\mu \rightarrowtail \mu^*$ (that is, $(\mu^*)^* = \mu$, for each μ in $\mathcal{A}$).

The two above multiplications distribute over addition. Also, $\mathcal{A}$ endowed with the following norm is a Banach algebra:

$$\|\beta\| = |\beta|\|\mu\|,$$
$$\|\mu + \varsigma\| \leq \|\mu\| + \|\varsigma\|,$$
$$\|\mu\varsigma\| \leq \|\mu\|\|\varsigma\| \text{ for all } \mu\varsigma \in \mathcal{A} \text{ and } \beta \in \mathbb{C}.$$

$\mathcal{A}$ is complete ($d(\mu, \varsigma) = \|\mu - \varsigma\|$). Finally, for each μ in $\mathcal{A}$, we have $\|\mu^*\mu\| = \|\mu\|^2$. $\mathcal{A}$ has an algebraic structure and a topological structure coming from a norm. The condition that $\mathcal{A}$ be a Banach algebra expresses a compatibility between these structures. For more synthesis on the work done in C^*-algebra valued metric spaces with some noteworthy remarks, we refer to [**5, 18, 19, 20, 21, 23, 24, 25, 26**].
Throughout the paper, $\mathcal{A}$ is an unital C^*-algebra with a unit $1_\mathcal{A}$, $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^+$ is the set of non-negative real numbers. $M_n(\mathbb{R})$ is $n \times n$ matrix with real entries.

Lemma 1.1. [**22**] *Let $\mathcal{A}$ be an unital C^*-algebra with a unit $1_\mathcal{A}$.*

(1) *For any $\Omega \in \mathcal{A}_+$, we have $\Omega \preceq 1_\mathcal{A}$ if and only if $\|\Omega\| \leq 1$.*
(2) *If $\mu \in \mathcal{A}_+$ with $\|\mu\| < \frac{1}{2}$, then $1_\mathcal{A} - \mu$ is invertible and $\|\mu(1_\mathcal{A} - \mu)^{-1}\| < 1$.*
(3) *If $\mu, \varsigma \in \mathcal{A}$ with $\mu, \varsigma \succeq 0_\mathcal{A}$ and $\mu\varsigma = \varsigma\mu$, then $\mu, \varsigma \succeq 0_\mathcal{A}$.*
(4) *Denote by $\mathcal{A}'$, the set $\{\mu \in \mathcal{A} : \mu\varsigma = \varsigma\mu, \forall \varsigma \in \mathcal{A}\}$. Let $\mu \in \mathbb{A}'$. If $\varsigma, \upsilon \in \mathbb{A}$ with $\varsigma \succeq \upsilon \succeq 0_\mathcal{A}$, and $1_\mathcal{A} - \mu \in \mathcal{A}'$ is an invertible operator, then*

$$(1_\mathcal{A} - \mu)^{-1}\varsigma \succeq (1_\mathcal{A} - \mu)^{-1}\upsilon.$$

In 2014, Ma and Jiang [**21**] introduced the concept of C^*-algebra-valued metric spaces as a new concept more general than metric spaces, by replacing the set of real numbers with C^*-algebras.

Definition 1.1. [**21**] *Let X be a non empty set. If $d : X \times X \to \mathcal{A}$ is such that*
(1.1)$0_\mathcal{A} \preceq d(\Omega, \tau)$;
(1.2)$d(\Omega, \tau) = 0_\mathcal{A}$ *if and only if* $\Omega = \tau$;
(1.3)$d(\Omega, \tau) = d(\tau, \Omega)$;
(1.4)$d(\Omega, \tau) \preceq d(\Omega, \omega) + d(\omega, \tau)$,
for all $\Omega, \tau, \omega \in X$, then d is called a C^*-algebra-valued metric on X, and $(X, \mathcal{A}, d)$ is called a C^*-algebra-valued metric space.

Definition 1.2. [**21**] *Let $(X, \mathcal{A}, d)$ be a C^*-algebra-valued metric space. The mapping $T : X \to X$ is called C^*-algebra-valued contraction if there is $P \in \mathcal{A}$ with $\|P\| < 1$ so that $d(T\Omega, T\tau) \preceq P^*d(\Omega, \tau)P$ for all $\Omega, \tau \in X$.*

Example 1.3. [**21**] Let $X = \mathbb{R}$ and $A = M_2(\mathbb{R})$. Take $d(\Omega, \tau) =$ diag $(|\Omega - \tau|, \alpha|\Omega - \tau|)$ where $\Omega, \tau \in \mathbb{R}$ and $\alpha \geqslant 0$ is a constant. Here, $(X, M_2(\mathbb{R}), d)$ is a complete C^*-algebra-valued metric space (follows from the completeness of $\mathbb{R}$). For more details, one can refer to [**21, 22, 23, 25, 26**].

2. Properties of compatible maps and its variants

In mathematics and theoretical physics in general, community is one of the basic concepts. Because it is important to know when and whether an operation with vectors, tensors, matrices and operators in general is commutative: $A * B = B * A$. Quantity is actually based on that notion. Thus, even with ordinary mappings, it is important to know whether they switch or not. The functions sin and cos do not commute because it is not true that $\sin(\cos x) = \cos(\sin x)$. Community is therefore a strong property. The first step in its corruption is consent. In the last century, G. Jungck [6]–[8],[9]–[12],[13]–[15],[17], introduced this more general term than community. This was done over strings. It is easy to check that two community mappings are compatible but that the reverse is not true. There is an even more general term, Jungck's weak consent. Which also has application in the theory of coincidence and point of coincidence.

We present variant notions of compatibility and its variants in the class of C^*-algebra-valued metric spaces.

Definition 2.1. *Let f and g be two self-mappings on a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$. Let $\{x_n\}$ be a sequence in X so that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$. Such f and g are said to be*

(1) compatible if $\lim_{n\to+\infty} d(fgx_n, gfx_n) = 0_{\mathcal{A}}$.

(2) compatible of type (A) if

$$\lim_{n\to+\infty} d(fgx_n, ggx_n) = 0_{\mathcal{A}} \quad \text{and} \quad \lim_{n\to+\infty} d(gfx_n, ffx_n) = 1_{\mathcal{A}}.$$

(3) compatible of type (B) if

$$\lim_{n\to+\infty} d(fgx_n, ggx_n) \preceq \frac{[\lim_{n\to+\infty} d(fgx_n, ft) + \lim_{n\to+\infty} d(ft, ffx_n)]}{2_{\mathcal{A}}}$$

and

$$\lim_{n\to+\infty} d(gfx_n, ffx_n) \preceq \frac{[\lim_{n\to+\infty} d(gfx_n, gt) + \lim_{n\to+\infty} d(gt, ggx_n)]}{2_{\mathcal{A}}}.$$

(4) compatible of type (C) if

$$\lim_{n\to+\infty} d(fgx_n, ggx_n) \preceq \frac{[\lim_{n\to+\infty} d(fgx_n, ft) + \lim_{n\to+\infty} d(ft, ffx_n) + \lim_{n\to+\infty} d(ft, ggx_n)]}{3_{\mathcal{A}}}$$

and

$$\lim_{n\to+\infty} d(gfx_n, ffx_n) \preceq \frac{\lim_{n\to+\infty} d(gfx_n, gt) + \lim_{n\to+\infty} d(gt, ggx_n) + \lim_{n\to+\infty} d(gt, ffx_n)]}{3_{\mathcal{A}}},$$

(5) compatible of type (P) if $\lim_{n\to+\infty} d(ffx_n, ggx_n) = 0_{\mathcal{A}}$.

In the following, we give the relationships and properties of above compatibilities.

Proposition 2.1. *Let f and g be two compatible mappings of type (A). If one of f and g is continuous, then f and g are compatible.*

Proof. Since f and g are compatible of type (A), we have
$0_{\mathcal{A}} = \lim_{n\to+\infty} d(fgx_n, ggx_n)$ and $0_{\mathcal{A}} = \lim_{n\to+\infty} d(gfx_n, ffx_n)$.

Suppose that f is continuous. Then $\lim_{n\to+\infty} ffx_n = \lim_{n\to+\infty} fgx_n = ft$ for some $t \in X$. We get $\lim_{n\to+\infty} d(fgx_n, gfx_n) = 0_{\mathcal{A}}$, i.e., f and g are compatible.

The case that g is continuous is done similarly. □

Proposition 2.2. *Every pair of compatible mappings of type (A) is compatible of type (B).*

Proof. Suppose that f and g are compatible of type (A). Then

$$\begin{aligned} 0_{\mathcal{A}} &= \lim_{n\to+\infty} d(fgx_n, ggx_n) \\ &\preceq \frac{\lim_{n\to+\infty} d(fgx_n, ft)\ + \lim_{n\to+\infty} d(ft, ffx_n)]}{2_{\mathcal{A}}} \end{aligned}$$

and

$$\begin{aligned} 0_{\mathcal{A}} &= \lim_{n\to+\infty} d(gfx_n, ffx_n) \\ &\preceq \frac{[\lim_{n\to+\infty} d(gfx_n, gt) + \lim_{n\to+\infty} d(gt, ggx_n)]}{2_{\mathcal{A}}}, \end{aligned}$$

as derived. □

Proposition 2.3. *Let f and g be continuous on a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$. If f and g are compatible of type (B), then they are compatible of type (A).*

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$. Since f and g are continuous, we have

$$\begin{aligned} \lim_{n\to+\infty} d(fgx_n, ggx_n) &\preceq \frac{[\lim_{n\to+\infty} d(fgx_n, ft) + \lim_{n\to+\infty} d(ft, ffx_n)]}{2_{\mathcal{A}}} \\ &= 0_{\mathcal{A}} \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to+\infty} d(gfx_n, ffx_n) &\preceq \frac{[\lim_{n\to+\infty} d(gfx_n, gt) + \lim_{n\to+\infty} d(gt, ggx_n)]}{2_{\mathcal{A}}} \\ &= 0_{\mathcal{A}}. \end{aligned}$$

Therefore, f and g are compatible of type (A). □

Proposition 2.4. *Let f and g be continuous on a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$ into itself. If f and g are compatible of type (B), then they are compatible.*

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$. Since f and g are continuous, we have

$$\lim_{n\to+\infty} ffx_n = ft = \lim_{n\to+\infty} fgx_n$$

and

$$\lim_{n\to+\infty} gfx_n = gt = \lim_{n\to+\infty} ggx_n.$$

By triangle inequality,

$$d(fgx_n, gfx_n) \preceq d(fgx_n, ggx_n) + d(ggx_n, gfx_n).$$

Letting $n \to +\infty$ and using f and g are compatible of type (B), we have

$$\begin{aligned}
&\lim_{n\to+\infty} d(fgx_n, gfx_n) \\
&\preceq \frac{\lim_{n\to+\infty} d(fgx_n, ft) + \lim_{n\to+\infty} d(ft, ffx_n)]}{2_{\mathcal{A}}} + \lim_{n\to+\infty} d(ggx_n, gfx_n) \\
&= 0_{\mathcal{A}}.
\end{aligned}$$

Therefore, f and g are compatible. □

Proposition 2.5. *Let f and g be continuous on a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$. If f and g are compatible, then they are compatible of type (B).*

Proof. Since f and g are compatible, there is $\{x_n\}$ a sequence in X so that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$ for which $\lim_{n\to+\infty} d(fgx_n, gfx_n) = 0_{\mathcal{A}}$. Since f and g are continuous,

$$\lim_{n\to+\infty} ffx_n = ft = \lim_{n\to+\infty} fgx_n$$

and

$$\lim_{n\to+\infty} gfx_n = gt = \lim_{n\to+\infty} ggx_n,$$

so

$$\lim_{n\to+\infty} ffx_n = \lim_{n\to+\infty} fgx_n = \lim_{n\to+\infty} gfx_n = \lim_{n\to+\infty} ggx_n.$$

Now

$$\lim_{n\to+\infty} d(fgx_n, ggx_n) \preceq \frac{[\lim_{n\to+\infty} d(fgx_n, ft) + \lim_{n\to+\infty} d(ft, ffx_n)]}{2_{\mathcal{A}}}$$

and

$$\lim_{n\to+\infty} d(gfx_n, ffx_n) \preceq \frac{[\lim_{n\to+\infty} d(gfx_n, gt) + \lim_{n\to+\infty} d(gt, ggx_n)]}{2_{\mathcal{A}}},$$

which give f and g be compatible of type (B). □

Proposition 2.6. *Let f and g be continuous on a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$. Then*

(1) *f and g are compatible if and only if they are compatible of type (B);*

(2) *f and g are compatible of type (A) if and only if they are compatible of type (B).*

Proof. (1) It suffices to use Propositions 2.4 and 2.5.

(2) It suffices to use Propositions 2.2 and 2.3. □

Proposition 2.7. *Let f and g be compatible mappings on a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$. If $ft = gt$ for some $t \in X$, then $fgt = fft = ggt = gft$.*

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n = t$, $n = 1, 2, \dots$ for some $t \in X$ and $ft = gt$. Then $fx_n, gx_n \to ft$ as $n \to +\infty$. Since f and g are compatible,

$$d(fgt, gft) = \lim_{n\to+\infty} d(fgx_n, gfx_n) = 0_{\mathcal{A}}.$$

Hence we have $fgt = ggt$. Since $ft = gt$, we have $fgt = fft = ggt = gft$. □

Utilizing Proposition 2.7, we infer

Proposition 2.8. *Let f and g be compatible mappings on a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$. Suppose that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$. Then*

(a) $\lim_{n\to+\infty} gfx_n = ft$ *if f is continuous at t.*

(b) $\lim_{n\to+\infty} fgx_n = gt$ *if g is continuous at t.*

(c) $fgt = gft$ *and* $ft = gt$ *if f and g are continuous at t.*

Proof. (a) Suppose that f is continuous at t. Since $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$, we have $fgx_n \to ft$ as $n \to +\infty$. The mappings f and g are compatible, so

$$\begin{aligned}\lim_{n\to+\infty} d(gfx_n, ft) &= \lim_{n\to+\infty} d(gfx_n, fgx_n) + \lim_{n\to+\infty} d(fgx_n, ft)\\ &= 0_{\mathcal{A}}.\end{aligned}$$

Therefore, $\lim_{n\to+\infty} gfx_n = ft$.

(b) The proof of $\lim_{n\to+\infty} fgx_n = gt$ follows by similar arguments as in (a).

(c) Suppose that f and g are continuous at t. Since $gx_n \to t$ as $n \to +\infty$ and f is continuous at t, by (a), $gfx_n \to ft$ as $n \to +\infty$. On the other hand, g is also continuous at t, $gfx_n \to gt$. Thus, we have $ft = gt$ by the uniqueness of limit and so by Proposition 2.8, $fgt = gft$. □

Proposition 2.9. *Let f and g be compatible mappings of type (B) on a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$. If $ft = gt$ for some $t \in X$, then $fgt = fft = ggt = gft$.*

Proof. Let $\{x_n\}$ be in X defined by $x_n =$ for $n = 1, 2, \dots$ (for some $t \in X$ with $ft = gt$). Then $fx_n, gx_n \to ft$ as $n \to +\infty$. Since f and g are compatible of type (B), we have

$$\begin{aligned} d(fgt, ggt) &= \lim_{n\to+\infty} d(fgx_n, ggx_n) \\ &\preceq \frac{[\lim_{n\to+\infty} d(fgx_n, fft) + \lim_{n\to+\infty} d(fft, ffx_n)]}{2_{\mathcal{A}}} \\ &= 0_{\mathcal{A}}. \end{aligned}$$

Thus, $fgt = ggt$. Since $ft = gt$, we have $fgt = fft = ggt = gft$. □

From Proposition 2.9, we have

Proposition 2.10. *Let f and g be compatible mappings of type (B) on a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$. Suppose that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$. Then*
(a) $\lim_{n\to+\infty} ggx_n = ft$ if f is continuous at t.
(b) $\lim_{n\to+\infty} ffx_n = gt$ if g is continuous at t.
(c) $fgt = gft$ and $ft = gt$ if f and g are continuous at t.

Proof. (a) Suppose that f is continuous at t. Since $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$, we have $ffx_n, fgx_n \to ft$ as $n \to +\infty$. Since f and g are compatible of type (B), we have

$$\begin{aligned} \lim_{n\to+\infty} d(ft, ggx_n) &= \lim_{n\to+\infty} d(fgx_n, ggx_n) \\ &\preceq \frac{[\lim_{n\to+\infty} d(fgx_n, ft) + \lim_{n\to+\infty} d(ft, ffx_n)]}{2_{\mathcal{A}}} \\ &= d(ft, ft) = 0_{\mathcal{A}}. \end{aligned}$$

Therefore, $\lim_{n\to+\infty} ggx_n = ft$.

(b) The proof of $\lim_{n\to+\infty} ffx_n = gt$ follows by similar arguments as in (a).

(c) Suppose that f and g are continuous at t. Since $gx_n \to t$ as $n \to +\infty$ and f is continuous at t, by (a), $ggx_n \to ft$ as $n \to +\infty$. On the other hand, g is also continuous at t, $ggx_n \to gt$. Thus, we have $ft = gt$ by the uniqueness of limit and so by Proposition 2.9, $fgt = gft$. □

Remark 2.1. In Proposition 2.10, let f and g be compatible maps of type (C) or of type (P) instead of of type (B), the conclusion of Proposition 2.10 remains the same.

Remark 2.2. In Proposition 2.10, let f and g be compatible maps of type (C) or of type (P) instead of type (B), the conclusion of Proposition 2.10 remains the same.

The notions of compatibilities and its variants are independent of each other.

Example 2.2. Take $X = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{C})$. Define $d : X \times X \to \mathcal{A}$ by

$$d(\nu, \mu) = \begin{pmatrix} |\nu - \mu| & 0 \\ 0 & k|\nu - \nu| \end{pmatrix},$$

where $k > 0$. Then $(X, \mathcal{A}, d)$ is a C^*-algebra-valued metric space . Consider f and g defined on X by

$$f\nu = \begin{cases} \frac{1}{\nu^2} & \text{if } \nu \neq 0, \\ 2 & \text{if } \nu = 0 \end{cases} \quad \text{and} \quad g\nu = \begin{cases} \frac{1}{\nu} & \text{if } \nu \neq 0, \\ 2 & \text{if } \nu = 0. \end{cases}$$

Then f and g are not continuous at $t = 0$. Define $\{x_n\}$ in X by $x_n = n$ for each $n = 1, 2, \ldots$. If $n \to +\infty$, we have $fx_n = \frac{1}{n^2} \to t = 0, \quad gx_n = \frac{1}{n} \to t = 0$ and

$$\lim_{n\to+\infty} d(fgx_n, gfx_n) = \lim_{n\to+\infty} d\left(f\frac{1}{n}, g\frac{1}{n^2}\right) = \lim_{n\to+\infty} d(n^2, n^2) \xrightarrow{\|\cdot\|} 0_{\mathcal{A}}.$$

While, we have no existence of the following limits:

$$\lim_{n\to+\infty} d(fgx_n, ggx_n) = \lim_{n\to+\infty} d(n^2, n) = \begin{pmatrix} |n^2 - n| & 0 \\ 0 & k|n^2 - n| \end{pmatrix} \xrightarrow{\|\cdot\|} +\infty.$$

$$\frac{\left[\lim_{n\to+\infty} d(fgx_n, f0) + \lim_{n\to+\infty} d(f0, ffx_n)\right]}{2_{\mathcal{A}}}$$
$$\xrightarrow{\|\cdot\|} +\infty$$

and

$$\lim_{n\to+\infty} d(gfx_n, ffx_n) \xrightarrow{\|\cdot\|} +\infty$$

$$\frac{\left[\lim_{n\to+\infty} d(gfx_n, g0) + \lim_{n\to+\infty} d(g0, ggx_n)\right]}{2_{\mathcal{A}}}$$
$$\xrightarrow{\|\cdot\|} +\infty$$

Also $\lim_{n\to+\infty} d(fgx_n, ggx_n) \xrightarrow{\|\cdot\|} +\infty$ and $\lim_{n\to+\infty} d(gfx_n, ffx_n) \xrightarrow{\|\cdot\|} +\infty$ and we get

$$\frac{\left[\lim_{n\to+\infty} d(fgx_n, f0) + \lim_{n\to+\infty} d(f0, ffx_n) + \lim_{n\to+\infty} d(f0, ggx_n)\right]}{3_{\mathcal{A}}}$$
$$\xrightarrow{\|\cdot\|} +\infty$$

and

$$\frac{\left[\lim_{n\to+\infty} d(gfx_n, g0) + \lim_{n\to+\infty} d(g0, ggx_n) + \lim_{n\to+\infty} d(g0, ffx_n)\right]}{3_{\mathcal{A}}}$$
$$\xrightarrow{\|\cdot\|} +\infty$$

Also

$$\lim_{n\to+\infty} d(ffx_n, ggx_n) = \lim_{n\to+\infty} d(n^4, n) \xrightarrow{\|\cdot\|} +\infty.$$

Therefore, f and g are compatible, but they are not compatible of type (A), compatible of type (B), type (C) and of type (P).

Example 2.3. Let $X = [2,4]$ and $\mathcal{A} = M_2(\mathbb{C})$. Let $d : X \times X \to \mathcal{A}$ by

$$d(\nu,\mu) = \begin{pmatrix} |\nu-\mu| & 0 \\ 0 & k|\nu-\mu| \end{pmatrix},$$

where $k > 0$. Then $(X, \mathcal{A}, d)$ is a C^*-algebra-valued metric space. Define f and $g : X \to X$ by

$$f\nu = \begin{cases} 2 & \text{if } \ \nu = 2 \ or \ \nu > 3, \\ \nu+1 & \text{if } \ 2 < \nu \le 3 \end{cases} \quad \text{and} \quad g\nu = \begin{cases} 2 & \text{if } \nu = 2, \\ \frac{\nu+4}{2} & \text{if } 2 < \nu \le 3 \\ \frac{\nu+1}{2} & \text{if } \nu > 3. \end{cases}$$

Then f and g are not continuous at $t = 2$. We claim that f and g are not compatible, but they are compatible of type (A), of type (B), of type (C) and of type (P).
For this, let $\{x_n\} = 3 + \frac{1}{n} \subset [2,4]$ so that $fx_n, gx_n \to 2$.
Now,

$$\lim_{n\to+\infty} d(fgx_n, gfx_n) \overset{\|\cdot\|}{\not\longrightarrow} 0_{\mathcal{A}}.$$

Further, we have

$$\lim_{n\to+\infty} d(fgx_n, ggx_n) \xrightarrow{\|\cdot\|} 0_{\mathcal{A}},$$

$$\frac{\left[\lim_{n\to+\infty} d(fgx_n, f2) + \lim_{n\to+\infty} d(f2, ffx_n)\right]}{2_{\mathcal{A}}}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{k}{2} \end{pmatrix}$$

and

$$\lim_{n\to+\infty} d(gfx_n, ffx_n) \xrightarrow{\|\cdot\|} 0_{\mathcal{A}},$$

$$\frac{\left[\lim_{n\to+\infty} d(gfx_n, g2) + \lim_{n\to+\infty} d(g2, ggx_n)\right]}{2_{\mathcal{A}}}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{k}{2} \end{pmatrix}$$

Also $\lim_{n\to+\infty} d(fgx_n, ggx_n) = 0_{\mathcal{A}}$ and $\lim_{n\to+\infty} d(gfx_n, ffx_n) = 0_{\mathcal{A}}$ and we get

$$\frac{\left[\lim_{n\to+\infty} d(fgx_n, f2) + \lim_{n\to+\infty} d(f2, ffx_n) + \lim_{n\to+\infty} d(f2, ggx_n)\right]}{3_{\mathcal{A}}} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2k}{3} \end{pmatrix}$$

and

$$\frac{\Big[\lim_{n\to+\infty} d(gfx_n, g2) + \lim_{n\to+\infty} d(g2, ggx_n) + \lim_{n\to+\infty} d(g2, ffx_n)\Big]}{3_{\mathcal{A}}} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2k}{3} \end{pmatrix}$$

as $x_n \to 2$ and $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = 2$.
Also,

$$\lim_{n\to+\infty} d(ffx_n, ggx_n) \overset{\|\cdot\|}{\not\longrightarrow} 0_{\mathcal{A}}.$$

Thus, f and g are compatible mappings of type (A), of type (B), of type (C) but f and g are not compatible and compatible of type (P).

3. Common fixed point results

In this section, we prove some common fixed point theorems via compatibility in C^*-algebra-valued metric space $(X, \mathcal{A}, d)$.

Theorem 3.1. *Let A, B, S and T be self-mappings on a complete C^*-algebra-valued metric space $(X, \mathcal{A}, d)$ so that:*
(C_1) *$SX \subset BX$ and $TX \subset AX$;*
(C_2)

$$(d(Sx, Ty))^2 \preceq P^* M(x, y) P,$$

for all $x, y \in X$, where $P \in \mathcal{A}$ with $\|P\| < 1$, and

$$M(x,y) = \max\{d(Ax,Sx)\cdot d(By,Ty), d(Ax,Ty)\cdot d(By,Sx), d(By,Sx)\cdot d(By,Ty), (d(Ax,By))^2, d(Ax,Sx)\cdot d(By,Sx), d(Ax,By)\cdot d(By,Sx), d(Ax,By)\cdot d(By,Ty)\};$$

(C_3) *one of A, B, S and T is continuous.*
If (A, S) and (B, T) are compatible, then A, B, T and S possess a unique common fixed point.

Proof. Take $x_0 \in X$. Since $SX \subset BX$, there is $x_1 \in X$ so that $Sx_0 = Bx_1 = \upsilon_0$. Similarly, since $TX \subset AX$, there is $x_2 \in X$ so that $Tx_1 = Ax_2 = \upsilon_1$. Continuing in this direction, one can construct sequences so that

$$\upsilon_{2n} = Sx_{2n} = Bx_{2n+1}, \quad \upsilon_{2n+1} = Tx_{2n+1} = Ax_{2n+2}.$$

Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (C_2), we have

$$(d(\upsilon_{2n}, \upsilon_{2n+1}))^2 = d(Sx_{2n}, Tx_{2n+1}) \preceq P^* M(x_{2n}, x_{2n+1}) P,$$

where

$$M(x_{2n}, x_{2n+1})$$
$$= \max\{d(Ax_{2n}, Sx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}), d(Ax_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sx_{2n}),$$
$$d(Bx_{2n+1}, Sx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}), (d(Ax_{2n}, Bx_{2n+1}))^2, d(Ax_{2n}, Sx_{2n})\cdot$$
$$d(Bx_{2n+1}, Sx_{2n}), d(Ax_{2n}, Bx_{2n+1}) \cdot d(Bx_{2n+1}, Sx_{2n}), d(Ax_{2n}, Bx_{2n+1})\cdot$$
$$d(Bx_{2n+1}, Tx_{2n+1})\} = \max\{d(\upsilon_{2n-1}, \upsilon_{2n}) \cdot d(\upsilon_{2n}, \upsilon_{2n+1}), d(\upsilon_{2n-1}, \upsilon_{2n+1}) \cdot d(\upsilon_{2n}, \upsilon_{2n}),$$
$$d(\upsilon_{2n}, \upsilon_{2n}) \cdot d(\upsilon_{2n}, \upsilon_{2n+1}), (d(\upsilon_{2n-1}, \upsilon_{2n}))^2, d(\upsilon_{2n-1}, \upsilon_{2n}) \cdot d(\upsilon_{2n}, \upsilon_{2n}),$$
$$d(\upsilon_{2n-1}, \upsilon_{2n}) \cdot d(\upsilon_{2n}, \upsilon_{2n}), d(\upsilon_{2n-1}, \upsilon_{2n}) \cdot d(\upsilon_{2n}, \upsilon_{2n+1})\}$$
$$= \max\{d(\upsilon_{2n-1}, \upsilon_{2n}) \cdot d(\upsilon_{2n}, \upsilon_{2n+1}), 0_{\mathcal{A}}, 0_{\mathcal{A}}, (d(\upsilon_{2n-1}, \upsilon_{2n}))^2, 0_{\mathcal{A}}, 0_{\mathcal{A}}, d(\upsilon_{2n-1}, \upsilon_{2n})\cdot$$
$$d(\upsilon_{2n}, \upsilon_{2n+1})\} = \max\{d(\upsilon_{2n-1}, \upsilon_{2n}) \cdot d(\upsilon_{2n}, \upsilon_{2n+1}), 0_{\mathcal{A}}, (d(\upsilon_{2n-1}, \upsilon_{2n}))^2\}.$$

Suppose that $d(\upsilon_{2n}, \upsilon_{2n+1}) \succ d(\upsilon_{2n-1}, \upsilon_{2n})$ for some n, then $(d(\upsilon_{2n}, \upsilon_{2n+1}))^2 \prec (d(\upsilon_{2n}, \upsilon_{2n+1}))^2$, a contradiction. So, $d(\upsilon_{2n}, \upsilon_{2n+1}) \preceq d(\upsilon_{2n-1}, \upsilon_{2n})$ for each $n \geq 1$. Hence

$$(d(\upsilon_{2n}, \upsilon_{2n+1}))^2 \preceq P^*(d(\upsilon_{2n-1}, \upsilon_{2n}))^2 P. \tag{1}$$

We also obtain

$$(d(\upsilon_{2n+1}, \upsilon_{2n+2}))^2 \preceq P^*(d(\upsilon_{2n}, \upsilon_{2n+1}))^2 P. \tag{2}$$

From (1) and (2), continuing in this direction,

$$(d(\upsilon_n, \upsilon_{n+1}))^2 \preceq \cdots \preceq (P^*)^n (d(\upsilon_0, \upsilon_1))^2 P^n,$$

for all $n \geq 2$. For $m, n \in \mathbb{N}$ with $m > n$, using triangular inequality in C^*-algebra-valued metric space $(X, \mathcal{A}, d)$, we have

$$(d(\upsilon_n, \upsilon_m))^2 \preceq (d(\upsilon_n, \upsilon_{n+1}))^2 + \cdots + (d(\upsilon_{m-1}, \upsilon_m))^2$$

because that d^2 is a C^*-algebra-valued metric as d is a C^*-algebra-valued metric. Thus,

$$\begin{aligned}(d(\upsilon_n,\upsilon_m))^2 &\preceq (P^*)^n(d(\upsilon_0,\upsilon_1))^2P^n+\cdots+(P^*)^{m-1}(d(\upsilon_0,\upsilon_1))^2P^{m-1}\\ &\preceq [(P^*)^nP^n+\ldots(P^*)^{m-1}P^{m-1}](d(\upsilon_0,\upsilon_1))^2\\ &\preceq [(P^n)^*P^n+\ldots(P^{m-1})^*P^{m-1}](d(\upsilon_0,\upsilon_1))^2\\ &\preceq \sum_{i=n}^{m-1}|P^i|^2(d(\upsilon_0,\upsilon_1))^2\\ &\preceq \|\sum_{i=n}^{m-1}|P^i|^2(d(\upsilon_0,\upsilon_1))^2\|1_{\mathcal{A}}\\ &\preceq \|\sum_{i=n}^{m-1}|P^i|^2\|\|(d(\upsilon_0,\upsilon_1))^2\|1_{\mathcal{A}}\\ &\preceq \sum_{i=n}^{m-1}\|P^{2i}\|\|(d(\upsilon_0,\upsilon_1))^2\|1_{\mathcal{A}}\\ &\preceq \frac{\|P\|^{2m-2}}{(1-\|P\|)}\|(d(\upsilon_0,\upsilon_1))^2\|1_{\mathcal{A}}.\end{aligned}$$

Thus, $d(\upsilon_n,\upsilon_m)^2$ tends to $0_{\mathcal{A}}$ as $m,n\to+\infty$. Thus, the sequence $\{\upsilon_n\}$ is Cauchy in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of the sequence $\{\upsilon_n\}$ also converge to z.

Suppose that A is continuous. Then $\{AAx_{2n}\}$, $\{ASx_{2n}\}$ converge to Az. Since A and S are compatible, by Proposition 2.8, $\{SAx_{2n}\}$ is convergent to Az. We claim that $z=Az$. Taking $x=Ax_{2n}$ and $y=x_{2n+1}$ in (C_2), we have

$$(d(SAx_{2n},Tx_{2n+1}))^2\preceq P^*M(Ax_{2n},x_{2n+1})P,$$

where

$$\begin{aligned}&M(Ax_{2n},x_{2n+1})=\max\{d(AAx_{2n},SAx_{2n})\cdot d(Bx_{2n+1},Tx_{2n+1}),d(AAx_{2n},Tx_{2n+1})\cdot\\ &d(Bx_{2n+1},SAx_{2n}),d(Bx_{2n+1},SAx_{2n})\cdot d(Bx_{2n+1},Tx_{2n+1}),(d(AAx_{2n},Bx_{2n+1}))^2,\\ &d(AAx_{2n},SAx_{2n})\cdot d(Bx_{2n+1},SAx_{2n}),d(AAx_{2n},Bx_{2n+1})\cdot d(Bx_{2n+1},SAx_{2n}),\\ &d(AAx_{2n},Bx_{2n+1})\cdot d(Bx_{2n+1},Tx_{2n+1})\}.\end{aligned}$$

Letting $n\to+\infty$, we have

$$(d(Az,z))^2\preceq P^*M(z,z)P,$$

where

$$\begin{aligned}M(z,z)=\max\{&d(Az,Az)\cdot d(z,z),d(Az,z)\cdot d(z,Az),d(z,Az)\cdot d(z,z),(d(Az,z))^2,\\ &d(Az,Az)\cdot d(z,Az),d(Az,z)\cdot d(z,Az),d(Az,z)\cdot d(z,z)\}.\end{aligned}$$

This gives that $d(Az,z)=0_{\mathcal{A}}$, i.e., $Az=z$.

We shall show that $Sz=z$. Consider $x=z$ and $y=x_{2n+1}$ in (C_2),

$$(d(Sz,Tx_{2n+1}))^2 \preceq P^*M(z,x_{2n+1})P,$$

where

$$M(z,x_{2n+1})=\max\{d(Az,Sz)\cdot d(Bx_{2n+1},Tx_{2n+1}), d(Az,Tx_{2n+1})\cdot d(Bx_{2n+1},Sz),$$
$$d(Bx_{2n+1},Sz)\cdot d(Bx_{2n+1},Tx_{2n+1}), (d(Az,Bx_{2n+1}))^2, d(Az,Sz)\cdot d(Bx_{2n+1},Sz),$$
$$d(Az,Bx_{2n+1})\cdot d(Bx_{2n+1},Sz), d(Az,Bx_{2n+1})\cdot d(Bx_{2n+1},Tx_{2n+1})\}.$$

Letting $n\to+\infty$, we have

$$(d(Sz,z))^2 \preceq P^*M(z,z)P,$$

where

$$M(z,z)=\max\{d(z,Sz)\cdot d(z,z), d(Az,z)\cdot d(z,Sz), d(z,Sz)\cdot d(z,z), (d(Az,z))^2,$$
$$d(z,Sz)\cdot d(z,Sz), d(z,z)\cdot d(z,Sz), d(z,z)\cdot d(z,z)\}.$$

This leads to $Sz=z$.
Since $SX\subset BX$, there is $u\in X$ so that $z=Sz=Bu$.

Now, putting $x=z$ and $y=u$ in (C_2),

$$(d(z,Tu))^2=(d(Sz,Tu))^2 \preceq P^*M(z,u)P,$$

where

$$M(z,u)=\max\{d(Az,Sz)\cdot d(Bu,Tu), d(Az,Tu)\cdot d(Bu,Sz), d(Bu,Sz)\cdot$$
$$d(Bu,Tu),(d(Az,Bu))^2, d(Az,Sz)\cdot d(Bu,Sz), d(Az,Bu)\cdot d(Bu,Sz), d(Az,Bu)\cdot$$
$$d(Bu,Tu)\}=\max\{d(z,z)\cdot d(z,z), d(z,z)\cdot d(z,z), d(z,z)\cdot d(z,Tu), (d(z,z))^2,$$
$$d(z,z)\cdot d(z,z), d(z,z)\cdot d(z,z), d(z,z)\cdot d(z,Tu)\}.$$

Thus, $z=Tu$. Since (B,T) is compatible and $Bu=Tu=z$, by Proposition 2.7, we get $BTu=TBu$ and so $Bz=BTu=TBu=Tz$. Putting again $x=z$ and $y=z$ in (C_2),

$$(d(z,Bz))^2=(d(Sz,Tz))^2 \preceq P^*M(z,z)P,$$

where

$$M(z,z)=\max\{d(Az,Sz)\cdot d(Bz,Tz), d(Az,Tz)\cdot d(Bz,Sz), d(Bz,Sz)\cdot d(Bz,Tz),$$
$$(d(Az,Bz))^2, d(Az,Sz)\cdot d(Bz,Sz), d(Az,Bz)\cdot d(Bz,Sz), d(Az,Bz)\cdot d(Bz,Tz)\}$$
$$=\max\{d(z,z)\cdot d(Bz,Tz), d(z,Bz)\cdot d(Bz,z), d(Bz,z)\cdot d(Bz,Tz),$$
$$(d(z,Bz))^2, d(z,z)\cdot d(Bz,z), d(z,Bz)\cdot d(Bz,z), d(z,Bz)\cdot d(Bz,Tz)\}.$$

Hence $z=Bz$. We deduce that $z=Bz=Tz=Az=Sz$, i.e., z is a common fixed point of A,B,S and T.

The proof is done similarly when B is continuous.

We consider that S is continuous. Here, $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ are convergent to Az as $n \to +\infty$. Since A and S are compatible, by Proposition 2.8, $\{ASx_{2n}\}$ is convergent to Az. Taking $x = Sx_{2n}$ and $y = x_{2n+1}$ in (C_2),

$$(d(SSx_{2n}, Tx_{2n+1}))^2 \preceq P^* M(Sx_{2n}, x_{2n+1})P,$$

where

$$M(Sx_{2n}, x_{2n+1}) = \max\{d(ASx_{2n}, SSx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}), d(ASx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SSx_{2n}), d(Bx_{2n+1}, SSx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}), (d(ASx_{2n}, Bx_{2n+1}))^2, d(ASx_{2n}, SSx_{2n}) \cdot d(Bx_{2n+1}, SSx_{2n}), d(ASx_{2n}, Bx_{2n+1}) \cdot d(Bx_{2n+1}, SSx_{2n}), d(ASx_{2n}, Bx_{2n+1}) \cdot d(Bx_{2n+1}, Tx_{2n+1})\}.$$

Letting $n \to +\infty$, we get

$$(d(Sz, z))^2 \preceq P^* M(z, z)P,$$

where

$$M(z, z) = \max\{d(Sz, Sz) \cdot d(z, z), d(Sz, z) \cdot d(z, Sz), d(z, Sz) \cdot d(z, z), (d(Sz, z))^2, d(Sz, Sz) \cdot d(z, Sz), d(Sz, z) \cdot d(z, Sz), d(Sz, z) \cdot d(z, z)\}.$$

so $Sz = z$. Since $SX \subset BX$, there is $v \in X$ so that $z = Sz = Bv$. By taking $x = Sx_{2n}$ and $y = v$ in (C_2), we have

$$(d(SSx_{2n}, Tv))^2 \preceq P^* M(Sx_{2n}, v)P,$$

where

$$M(Sx_{2n}, v) = \max\{d(ASx_{2n}, SSx_{2n}) \cdot d(Bv, Tv), d(ASx_{2n}, Tv) \cdot d(Bv, SSx_{2n}), d(Bv, SSx_{2n}) \cdot d(Bv, Tv), (d(ASx_{2n}, Bv))^2, d(ASx_{2n}, SSx_{2n}) \cdot d(Bv, SSx_{2n}), d(ASx_{2n}, Bv) \cdot d(Bv, SSx_{2n}), d(ASx_{2n}, Bv) \cdot d(Bv, Tv)\}.$$

Letting $n \to +\infty$, we have

$$(d(z, Tv))^2 \preceq P^* M(z, v)P,$$

where

$$M(z, v) = \max\{d(z, z) \cdot d(z, Tv), d(z, Tv) \cdot d(z, z), d(z, z) \cdot d(z, Tv), (d(z, z))^2, d(z, z) \cdot d(z, z), d(z, z) \cdot d(z, z), d(z, z) \cdot d(z, Tv)\},$$

that is, $z = Tv$.
Since B and T are compatible and $Bv = Tv = z$, by Proposition 2.7, we have $BTv = TBv$, so $Bz = BTv = TBv = Tz$. Consider

$$(d(Sx_{2n}, Tz))^2 \preceq P^* M(x_{2n}, z)P,$$

where

$$M(x_{2n},z)=\max\{d(Ax_{2n},Sx_{2n})\cdot d(Bz,Tz),d(Ax_{2n},Tz)\cdot d(Bz,Sx_{2n}),$$
$$d(Bz,Sx_{2n})\cdot d(Bz,Tz),(d(Ax_{2n},Bz))^2,d(Ax_{2n},Sx_{2n})\cdot d(Bz,Sx_{2n}),$$
$$d(Ax_{2n},Bz)\cdot d(Bz,Sx_{2n}),d(Ax_{2n},Bz)\cdot d(Bz,Tz)\}.$$

Letting $n\to+\infty$, we get

$$(d(z,Tz))^2\preceq P^*M(z,z)P,$$

where

$$M(z,z)=\max\{d(z,z)\cdot d(Bz,Tz),d(z,Tz)\cdot d(Tz,z),d(Tz,z)\cdot d(Bz,Tz),$$
$$(d(z,Tz))^2,d(z,z)\cdot d(Tz,z),d(z,Tz)\cdot d(Tz,z),d(z,Tz)\cdot d(Bz,Tz)\},$$

so $Tz=z$.
Since $TX\subset AX$, there is $w\in X$ so that $z=Tz=Aw$.

On taking $x=w$ and $y=z$ in (C_2), we get

$$(d(Sw,z))^2=(d(Sw,Tz))^2\preceq P^*M(w,z)P,$$

where

$$M(w,z)=\max\{d(Aw,Sw)\cdot d(Bz,Tz),d(Aw,Tz)\cdot d(Bz,Sw),d(Bz,Sw)\cdot$$
$$d(Bz,Tz),(d(Aw,Bz))^2,d(Aw,Sw)\cdot d(Bz,Sw),d(Aw,Bz)\cdot d(Bz,Sw),d(Aw,Bz)\cdot$$
$$d(Bz,Tz)\}=\max\{d(z,Sw)\cdot d(z,z),d(z,z)\cdot d(z,Sw),d(z,Sw)\cdot d(z,z),(d(z,z))^2,$$
$$d(z,Sw)\cdot d(z,Sw),d(z,z)\cdot d(z,Sw),d(z,z)\cdot d(z,z)\}.$$

Thus, $Sw=z$. Since A and S are compatible and $Sw=Aw=z$, by Proposition 2.7, we have $ASw=SAw$, so $Az=ASw=SAw=Sz$, i.e., $z=Az=Sz=Bz=Tz$. Therefore, z is a common fixed point of A,B,S and T.

The proof is similar when T is continuous.

Finally, suppose that z and w $(z\neq w)$ are two common fixed points of A,B,S and T.
On putting $x=z$ and $y=w$ in (C_2), we get

$$(d(z,w))^2=(d(Sz,Tw))^2\preceq P^*M(z,w)P,$$

where

$$M(z,w)=\max\{d(Az,Sz)\cdot d(Bw,Tw),d(Az,Tw)\cdot d(Bw,Sz),d(Bw,Sz)\cdot$$
$$d(Bw,Tw),(d(Az,Bw))^2,d(Az,Sz)\cdot d(Bw,Sz),d(Az,Bw)\cdot d(Bw,Sz),$$
$$d(Az,Bw)\cdot d(Bw,Tw)\}=\max\{d(z,z)\cdot d(w,w),d(z,w)\cdot d(w,z),d(w,z)\cdot d(w,w),$$
$$(d(z,w))^2,d(z,z)\cdot d(w,z),d(z,w)\cdot d(w,z),d(z,w)\cdot d(w,w)\},$$

which yields that $z=w$. Therefore, z is the unique common fixed point of A,B,S and T. □

The following corresponds to compatible mappings of type (A).

Theorem 3.2. *Let A, B, S and T be mappings of a complete C^*-algebra-valued metric space $(X, \mathcal{A}, d)$ into itself satisfying $(C1)$-$(C3)$. If (A, S) and (B, T) are compatible of type (A), then A, B, S and T have a unique common fixed point.*

Proof. Assume that A is continuous. Since (A, S) is compatible of type (A), by Proposition 2.1, (A, S) is compatible, so result easily follows using Theorem 3.1.

Similarly, if B is continuous and (B, T) is compatible of type (A), then (B, T) is compatible so result easily follows from Theorem 3.1.

Similarly, we can complete the proof when S or T is continuous. □

The following is for compatible mappings of type (B).

Theorem 3.3. *Let A, B, S and T be mappings of a complete C^*-algebra-valued metric space $(X, \mathcal{A}, d)$ into itself satisfying $(C1)$-$(C3)$. If (A, S) and (B, T) are compatible of type (B), then A, B, S and T have a unique common fixed point.*

Proof. From Theorem 3.1, we have that $\{\upsilon_n\}$ is a Cauchy sequence in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{\upsilon_n\}$ converge to z.

Assume that S is continuous.
Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converge to Sz as $n \to +\infty$. The compatibility of (A, S) of type (B), and Proposition 2.10 yield that $\{AAx_{2n}\}$ is convergent to Sz.

On putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C_2), we get

$$(d(SAx_{2n}, Tx_{2n+1}))^2 \preceq P^* M(Ax_{2n}, x_{2n+1})P,$$

where

$$\begin{aligned}&M(Ax_{2n}, x_{2n+1}) = \max\{d(AAx_{2n}, SAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}),\\ &d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n}), d(Bx_{2n+1}, SAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}),\\ &(d(AAx_{2n}, Bx_{2n+1}))^2, d(AAx_{2n}, SAx_{2n}) \cdot d(Bx_{2n+1}, SAx_{2n}), d(AAx_{2n}, Bx_{2n+1})\cdot\\ &d(Bx_{2n+1}, SAx_{2n}), d(AAx_{2n}, Bx_{2n+1}) \cdot d(Bx_{2n+1}, Tx_{2n+1})\}.\end{aligned}$$

Letting $n \to +\infty$, we get

$$(d(Sz, z))^2 \preceq P^* M(z, z)P,$$

where

$$\begin{aligned}M(z, z) = \max\{&d(Sz, Sz) \cdot d(z, z), d(Sz, z) \cdot d(z, Sz), d(z, Sz) \cdot d(z, z), (d(Sz, z))^2,\\ &d(Sz, Sz) \cdot d(z, Sz), d(Sz, z) \cdot d(z, Sz), d(Sz, z) \cdot d(z, z)\},\end{aligned}$$

That is, $Sz = z$. Since $SX \subset BX$, there is $u \in X$ so that $z = Sz = Bu$. Putting $x = Ax_{2n}$ and $y = u$ in (C_2), we get

$$(d(SAx_{2n}, Tu))^2 \preceq P^* M(Ax_{2n}, u)P,$$

where

$$M(Ax_{2n}, u) = \max\{d(AAx_{2n}, SAx_{2n}) \cdot d(Bu, Tu), d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n}), d(Bu, SAx_{2n}) \cdot d(Bu, Tu), (d(AAx_{2n}, Bu))^2, d(AAx_{2n}, SAx_{2n}) \cdot d(Bu, SAx_{2n}), d(AAx_{2n}, Bu) \cdot d(Bu, SAx_{2n}), d(AAx_{2n}, Bu) \cdot d(Bu, Tu)\}.$$

Letting $n \to +\infty$,

$$(d(Sz, Tu))^2 \preceq P^* M(z, u) P,$$

where

$$M(z, u) = \max\{d(Sz, Sz) \cdot d(Sz, Tu), d(Sz, Tu) \cdot d(Sz, Sz), d(Sz, Sz) \cdot d(Sz, Tu), (d(Sz, Sz))^2, d(Sz, Sz) \cdot d(Sz, Sz), d(Sz, Sz) \cdot d(Sz, Sz), d(Sz, Sz) \cdot d(Sz, Tu)\}.$$

This gives that $Tu = Sz$ $(z = Tu)$. Since the pair (B, T) is compatible of type (B) and $Bu = z = Tu$. By Proposition 2.10, we have $TBu = BTu$ and so $Bz = BTu = TBu = Tz$.
On taking $x = x_{2n}$ and $y = z$ in (C_2),

$$(d(Sx_{2n}, Tz))^2 \preceq P^* M(x_{2n}, z) P,$$

where

$$M(x_{2n}, z) = \max\{d(Ax_{2n}, Sx_{2n}) \cdot d(Bz, Tz), d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n}), d(Bz, Sx_{2n}) \cdot d(Bz, Tz), (d(Ax_{2n}, Bz))^2, d(Ax_{2n}, Sx_{2n}) \cdot d(Bz, Sx_{2n}), d(Ax_{2n}, Bz) \cdot d(Bz, Sx_{2n}), d(Ax_{2n}, Bz) \cdot d(Bz, Tz)\}.$$

Letting $n \to +\infty$,

$$(d(z, Tz))^2 \preceq P^* M(z, z) P,$$

where

$$M(z, z) = \max\{d(z, z) \cdot d(Tz, Tz), d(z, Tz) \cdot d(Tz, z), d(Tz, z) \cdot d(Tz, Tz), (d(z, Tz))^2, d(z, z) \cdot d(Tz, z), d(z, Tz) \cdot d(Tz, z), d(z, Tz) \cdot d(Tz, Tz)\},$$

which leads to $Tz = z$.
Since $TX \subset AX$, there is $v \in X$ so that $z = Tz = Av$. On putting $x = v$ and $y = z$ in (C_2), we get

$$(d(Sv, Tz))^2 \preceq P^* M(v, z) P,$$

where

$$M(v, z) = \max\{d(Av, Sv) \cdot d(Bz, Tz), d(Av, Tz) \cdot d(Bz, Sv), d(Bz, Sv) \cdot d(Bz, Tz), (d(Av, Bz))^2, d(Av, Sv) \cdot d(Bz, Sv), d(Av, Bz) \cdot d(Bz, Sv), d(Av, Bz) \cdot d(Bz, Tz)\},$$

so $Sv = z$. Since the pair (A, S) is compatible of type (B) and $Sv = z = Av$, it follows from Proposition 2.9 that $Sz = SAv = ASv = Az$. Therefore, $Az = Bz = Sz = Tz = z$ and so z is a common fixed point of A, B, S and T.

Suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converge to Az as $n \to +\infty$. Since (A, S) is compatible of type (B), it follows from Proposition 2.10 that $\{SSx_{2n}\}$ is convergent to Az. By putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in (C_2), we get

$$(d(SSx_{2n}, Tx_{2n+1}))^2 \preceq P^* M(Sx_{2n}, x_{2n+1})P,$$

where

$$\begin{aligned}&M(Sx_{2n}, x_{2n+1}) = \max\{d(ASx_{2n}, SSx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}), d(ASx_{2n}, Tx_{2n+1})\cdot \\ &d(Bx_{2n+1}, SSx_{2n}), d(Bx_{2n+1}, SSx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}), (d(ASx_{2n}, Bx_{2n+1}))^2, \\ &d(ASx_{2n}, SSx_{2n}) \cdot d(Bx_{2n+1}, SSx_{2n}), d(ASx_{2n}, Bx_{2n+1}) \cdot d(Bx_{2n+1}, SSx_{2n}), \\ &d(ASx_{2n}, Bx_{2n+1}) \cdot d(Bx_{2n+1}, Tx_{2n+1})\}\end{aligned}$$

Letting $n \to +\infty$, we get

$$(d(Az, z))^2 \preceq P^* M(z, z)P, \ \ for\ all\ x, y \in X,$$

where

$$\begin{aligned}&M(z, z) = \max\{d(Az, Az) \cdot d(z, z), d(Az, z) \cdot d(z, Az), d(z, Az) \cdot d(z, z), (d(Az, z))^2, \\ &d(Az, Az) \cdot d(z, Az), d(Az, z) \cdot d(z, Az), d(Az, z) \cdot d(z, z)\}.\end{aligned}$$

That is, $Az = z$. On putting $x = z$ and $y = x_{2n+1}$ in (C_2), we get

$$(d(Sz, Tx_{2n+1}))^2 \preceq P^* M(z, x_{2n+1})P,$$

where

$$\begin{aligned}&M(z, x_{2n+1}) = \max\{d(Az, Sz) \cdot d(Bx_{2n+1}, Tx_{2n+1}), d(Az, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sz), \\ &d(Bx_{2n+1}, Sz) \cdot d(Bx_{2n+1}, Tx_{2n+1}), (d(Az, Bx_{2n+1}))^2, d(Az, Sz) \cdot d(Bx_{2n+1}, Sz), \\ &d(Az, Bx_{2n+1}) \cdot d(Bx_{2n+1}, Sz), d(Az, Bx_{2n+1}) \cdot d(Bx_{2n+1}, Tx_{2n+1})\}.\end{aligned}$$

Letting $n \to +\infty$, we get

$$(d(Sz, z))^2 \preceq P^* M(z, z)P, \ \ for\ all\ x, y \in X,$$

where

$$\begin{aligned}&M(z, z) = \max\{d(z, Sz) \cdot d(z, z), d(z, z) \cdot d(z, Sz), d(z, Sz) \cdot d(z, z), (d(z, z))^2, \\ &d(z, Sz) \cdot d(z, Sz), d(z, z) \cdot d(z, Sz), d(z, z) \cdot d(z, z)\}.\end{aligned}$$

Hence $Sz = z$. Since $SX \subset BX$, there is $w \in X$ so that $z = Sz = Bw$. On putting $x = z$ and $y = w$ in (C_2), we get

$$(d(z, Tw))^2 = (d(Sz, Tw))^2 \preceq P^* M(z, w)P, \; for\, all\, x, y \in X,$$

where

$$M(z, w) = \max\{d(Az, Sz) \cdot d(Bu, Tw), d(Az, Tw) \cdot d(Bw, Sz), d(Bw, Sz) \cdot d(Bw, Tw), (d(Az, Bw))^2, d(Az, Sz) \cdot d(Bw, Sz), d(Az, Bw) \cdot d(Bw, Sz), d(Az, Bw) \cdot d(Bw, Tw)\}$$
$$= \max\{d(z, z) \cdot d(z, Tw), d(z, Tw) \cdot d(z, z), d(z, z) \cdot d(z, Tw), (d(z, z))^2, d(z, z) \cdot d(z, z), d(z, z) \cdot d(z, z), d(z, z) \cdot d(z, Tw)\}.$$

This leads to $z = Tw$. Since (B, T) is compatible of type (B) and $Bw = z = Tw$, from Proposition 2.9, $TBw = BTw$ and so $Bz = BTw = TBw = Tz$.
On taking $x = z$ and $y = z$ in (C_2), we have

$$(d(Sz, Tz))^2 \preceq P^* M(z, z)P,$$

where

$$M(z, z) = \max\{d(Az, Sz) \cdot d(Bz, Tz), d(Az, Tz) \cdot d(Bz, Sz), d(Bz, Sz) \cdot d(Bz, Tz), (d(Az, Bz))^2, d(Az, Sz) \cdot d(Bz, Sz), d(Az, Bzy) \cdot d(Bz, Sz), d(Az, Bz) \cdot d(Bz, Tz)\} = \max\{d(z, z) \cdot d(Tz, Tz), d(z, Tz) \cdot d(Tz, z), d(Tz, z) \cdot d(Tz, Tz), (d(z, Bz))^2, d(z, z) \cdot d(Bz, z), d(z, Tz) \cdot d(Tz, z), d(z, Tz) \cdot d(Tz, Tz)\}.$$

Then $z = Tz$. Therefore, z is a common fixed point of A, B, S and T.

The proof is similar when B or T are continuous.

Finally, if z and w $(z \neq w)$ are two common fixed points, then we have

$$(d(z, w))^2 = (d(Sz, Tw))^2 \preceq P^* M(z, w)P,$$

where

$$M(z, w) = \max\{d(Az, Sz) \cdot d(Bw, Tw), d(Az, Tw) \cdot d(Bw, Sz), d(Bw, Sz) \cdot d(Bw, Tw), (d(Az, Bw))^2, d(Az, Sz) \cdot d(Bw, Sz), d(Az, Bw) \cdot d(Bw, Sz), d(Az, Bw) \cdot d(Bw, Tw)\},$$

so $z = w$. Therefore, z is the unique common fixed point of A, B, S and T.

□

The following corresponds to compatible mappings of type (C).

Theorem 3.4. *Let A, B, S and T be mappings of a C^*-algebra-valued metric space $(X, \mathcal{A}, d)$ into itself satisfying $(C1)$-$(C3)$.*

Suppose that the pairs (A, S) and (B, T) are compatible of type (C). Then A, B, S and T have a unique common fixed point.

PROOF. Here we skip the proof, keeping in view the length of the paper. □

Finally, we give the following theorem for compatible mappings of type (P).

Theorem 3.5. *Let A, B, S and T be mappings of a complete C^*-algebra-valued metric space $(X, \mathcal{A}, d)$ into itself satisfying $(C1)$-$(C3)$.*

Suppose that the pairs (A, S) and (B, T) are compatible of type (P). Then A, B, S and T have a unique common fixed point.

PROOF. From Theorem 3.1, we have that $\{\upsilon_n\}$ is a Cauchy sequence in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{\upsilon_n\}$ converge to z as $n \to +\infty$.

Assume that S is continuous.

Then $\{SSx_{2n}\}, \{SAx_{2n}\}$ converge to Sz as $n \to +\infty$. Since (A, S) is compatible of type (P), $\{AAx_{2n}\}$ converges to Sz as $n \to +\infty$.

We claim that $Sz = z$. On putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C_2), we have

$$(d(SAx_{2n}, Tx_{2n+1}))^2 \preceq P^* M(Ax_{2n}, x_{2n+1})P,$$

where

$$\begin{aligned}&M(Ax_{2n}, x_{2n+1}) = \max\{d(AAx_{2n}, SAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}),\\&d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n}), d(Bx_{2n+1}, SAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1}),\\&(d(AAx_{2n}, Bx_{2n+1}))^2, d(AAx_{2n}, SAx_{2n}) \cdot d(Bx_{2n+1}, SAx_{2n}), d(AAx_{2n}, Bx_{2n+1})\cdot\\&d(Bx_{2n+1}, SAx_{2n}), d(AAx_{2n}, Bx_{2n+1}) \cdot d(Bx_{2n+1}, Tx_{2n+1})\}.\end{aligned}$$

Letting $n \to +\infty$, we have

$$(d(Sz, z))^2 \preceq P^* M(z, z)P,$$

where

$$\begin{aligned}M(z, z) = \max\{&d(Sz, Sz) \cdot d(z, z), d(Sz, z) \cdot d(z, Sz), d(z, Sz) \cdot d(z, z),\\&(d(Sz, z))^2, d(Sz, Sz) \cdot d(z, Sz), d(Sz, z) \cdot d(z, Sz), d(Sz, z) \cdot d(z, z)\}.\end{aligned}$$

This proves that $Sz = z$. Since $SX \subset BX$, there is $u \in X$ so that $z = Sz = Bu$.

We claim that $Tu = z$. On taking $x = x_{2n}$ and $y = u$ in (C_2), we have

$$(d(Sx_{2n}, Tu))^2 \preceq P^* M(x_{2n}, u)P,$$

where

$$\begin{aligned}&M(x_{2n}, u) = \max\{d(Ax_{2n}, Sx_{2n}) \cdot d(Bu, Tu), d(Ax_{2n}, Tu) \cdot d(Bu, Sx_{2n}),\\&d(Bu, Sx_{2n}) \cdot d(Bu, Tu), (d(Ax_{2n}, Bu))^2, d(Ax_{2n}, Sx_{2n}) \cdot d(Bu, Sx_{2n}),\\&d(Ax_{2n}, Bu) \cdot d(Bu, Sx_{2n}), d(Ax_{2n}, Bu) \cdot d(Bu, Tu)\}.\end{aligned}$$

At the limit $n \to +\infty$, we have

$$(d(z, Tu))^2 \preceq P^* M(z, u) P,$$

where

$$\begin{aligned} M(z,u) = \max\{&d(z,z)\cdot d(z,Tu), d(z,Tu)\cdot d(z,z), d(z,z)\cdot d(z,Tu),\\ &(d(z,z))^2, d(z,z)\cdot d(z,z), d(z,z)\cdot d(z,z), d(z,z)\cdot d(z,Tu)\}.\end{aligned}$$

Thus, $z = Tu$. So $Bu = Tu = z$. Since (B, T) is compatible of type (P), we have $TTu = BBu$, which gives that $d(Bz, Tz) = 0_{\mathcal{A}}$. Hence $Tz = Bz$.

Now we claim that $Tz = z$. On putting $x = x_{2n}$ and $y = z$ in (C_2), we have

$$(d(Sx_{2n}, Tz))^2 \preceq P^* M(x_{2n}, z) P,$$

where

$$\begin{aligned} &M(x_{2n}, z) = \max\{d(Ax_{2n}, Sx_{2n})\cdot d(Bz, Tz), d(Ax_{2n}, Tz)\cdot d(Bz, Sx_{2n}),\\ &d(Bz, Sx_{2n})\cdot d(Bz, Tz), (d(Ax_{2n}, Bz))^2, d(Ax_{2n}, Sx_{2n})\cdot d(Bz, Sx_{2n}),\\ &d(Ax_{2n}, Bz)\cdot d(Bz, Sx_{2n}), d(Ax_{2n}, Bz)\cdot d(Bz, Tz)\}.\end{aligned}$$

Letting $n \to +\infty$, we have $z = Tz$, so $Bz = Tz = z$.
Since $TX \subset AX$, there is $v \in X$ so that $z = Tz = Av$.

We claim that $Sv = z$. Now, on taking $x = v$ and $y = z$ in (C_2), we have

$$(d(Sv, z))^2 = (d(Sv, Tz))^2 \preceq P^* M(v, z) P,$$

where

$$\begin{aligned} &M(v,z) = \max\{d(Av, Sv)\cdot d(Bz, Tz), d(Av, Tz)\cdot d(Bz, Sv), d(Bz, Sv)\cdot\\ &d(Bz, Tz), (d(Av, Bz))^2, d(Av, Sv)\cdot d(Bz, Sv), d(Av, Bz)\cdot d(Bz, Sv), d(Av, Bz)\cdot\\ &d(Bz, Tz)\}.\end{aligned}$$

This implies that $z = Sv$. Therefore, $z = Sv = Av$. Since (A, S) is compatible of type (P), we have $SSv = AAv$, which implies that $d(Sz, Az) = 0_{\mathcal{A}}$. Hence $Sz = Az$. Since $Az = Bz = Sz = Tz = z$, z is a common fixed point of A, B, S and T.

The proof is the same when either A, or B, or T is continuous.

The uniqueness follows easily. □

Following example supports our main theorems, there by making our concepts more transparent and easy to understand.

Example 3.1. *Let $X = [0, \frac{3}{2}]$ with $\mathcal{A} = M_2(\mathbb{C})$.*
Define $d : X \times X \to \mathcal{A}$ by

$$d(\nu, \mu) = \begin{pmatrix} |\nu - \mu| & 0 \\ 0 & k|\nu - \mu| \end{pmatrix},$$

where $k > 0$. Clearly, $(X, \mathcal{A}, d)$ is a C^*-algebra-valued metric space .
Define $S\nu = 1$, $T\nu = 1$, $B\nu = \nu$ and $A\nu = \frac{\nu+1}{2}$ for all $\nu \geq 1$. Note that

(i) $SX \subset BX, TX \subset AX$;

(ii) A, B, S *and* T *are continuous;*

(iii) the pairs (A, S) *and* (B, T) *are compatible, and are compatible of type* (A), *of type* (B), *of type* (C) *and of type* (P).

Take $x_n = 1 + \frac{1}{n}$ *for* $n \geq 1$. *Then* $x_n \to 1$ *as* $n \to +\infty$. *Now,*

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Bx_n = \lim_{n\to+\infty} Tx_n = 1 = t \in X$$

as $n \to +\infty$. *Also,*

$$\lim_{n\to+\infty} d(ASx_n, SAx_n) = 0_{\mathcal{A}}, \quad \lim_{n\to+\infty} d(BTx_n, TBx_n) = 0_{\mathcal{A}},$$
$$\lim_{n\to+\infty} d(ASx_n, SSx_n) = 0_{\mathcal{A}}, \quad \lim_{n\to+\infty} d(SAx_n, AAx_n) = 0_{\mathcal{A}},$$
$$\lim_{n\to+\infty} d(BTx_n, TTx_n) = 0_{\mathcal{A}}, \quad \lim_{n\to+\infty} d(TBx_n, BBx_n) = 0_{\mathcal{A}}.$$

(iv) For $\|P\| < 1$, *we have*

$$(d(Sx, Ty))^2 \preceq P^* M(x, y) P,$$

holds for all $x, y \in X$, *where* $P \in \mathcal{A}$ *and*

$$M(x, y) = \max\{d(Ax, Sx) \cdot d(By, Ty), d(Ax, Ty) \cdot d(By, Sx), d(By, Sx) \cdot d(By, Ty),$$
$$(d(Ax, By))^2, d(Ax, Sx) \cdot d(By, Sx), d(Ax, By) \cdot d(By, Sx),$$
$$d(Ax, By) \cdot d(By, Ty)\}.$$

All conditions of main theorems are satisfied, and 1 *is the unique common fixed point of* A, B, S *and* T.

The interested reader could also consult [**3**, **4**].

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