Mathematical Analysis and its Contemporary Applications
Volume 3, Issue 2, 2021, 40–54
doi: 10.30495/MACA.2021.1929674.1010
ISSN 2716-9898

Some coupled coincidence point results in partially ordered metric spaces

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Abstract. In this paper, we introduce the notion of partial-compatibility of mappings in an ordered partial metric space and use this notion to establish coupled coincidence point theorems for ϕ-mixed monotone mappings satisfying a nonlinear contraction condition. Our consequences is an extension of the results of Shatanawi et al. [W. Shatanawi, B. Samet and M. Abbas, *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comp. Model., 55(3-4) (2012), 680-687]. We also provide an example to illustrate the results presented herein.

1. Introduction and preliminaries

The concepts of mixed monotone mapping and coupled fixed point have been introduced in [**5**] by Bhaskar and Lakshmikantham and they established some coupled fixed point theorems for a mixed monotone mapping in partially ordered metric spaces. Since then, coupled fixed point theory has been a subject of interest by many authors regarding the application potential of it, for example, see [**4**, **10**, **11**, **12**, **13**, **14**] and references therein.

Definition 1.1. [**5**] Let $(\Omega, \preceq)$ be a partially ordered set and $\Sigma : \Omega \times \Omega \to \Omega$ be a given map. We say that Σ has the mixed monotone property if $\Sigma(\alpha, \beta)$ is monotone nondecreasing in α and is monotone nonincreasing in β, that is, for all $\alpha_1, \alpha_2 \in \Omega$, $\alpha_1 \preceq \alpha_2$ implies $\Sigma(\alpha_1, \beta) \preceq \Sigma(\alpha_2, \beta)$ for any $\beta \in \Omega$, and for all $\beta_1, \beta_2 \in \Omega$, $\beta_1 \succeq \beta_2$ implies that $\Sigma(\alpha, \beta_1) \preceq \Sigma(\alpha, \beta_2)$, for any $\alpha \in \Omega$.

Definition 1.2. [**5**] An element $(\alpha, \beta) \in \Omega \times \Omega$ is called a coupled fixed point of mapping $\Sigma : \Omega \times \Omega \to \Omega$ if $\alpha = \Sigma(\alpha, \beta)$ and $\beta = \Sigma(\beta, \alpha)$.

2010 *Mathematics Subject Classification.* Primary: 47H10; Secondary: 54H25.
Key words and phrases. Coupled coincidence point, partially ordered set, partial metric space, mixed monotone property, compatible mapping.

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Later, Lakshmikantham and Ciric [17] introducing the notions of mixed ϕ-monotone mapping and a coupled coincidence point proved coupled coincidence and the common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces.

Definition 1.3. [17] Let $(\Omega, \preceq)$ be a partially ordered set and $\Sigma : \Omega \times \Omega \to \Omega$ and $\phi : \Omega \to \Omega$ be given two mappings. Σ has the mixed ϕ-monotone property if Σ is monotone ϕ-nondecreasing in its first argument and is monotone ϕ-nonincreasing in its second argument, that is, if for all $\alpha_1, \alpha_2 \in \Omega$, $\phi\alpha_1 \preceq \phi\alpha_2$ implies $\Sigma(\alpha_1, \beta) \preceq \Sigma(\alpha_2, \beta)$ for any $\beta \in \Omega$, and for all $\beta_1, \beta_2 \in \Omega$, $\phi\beta_1 \succeq \phi\beta_2$ implies $\Sigma(\alpha, \beta_1) \preceq \Sigma(\alpha, \beta_2)$ for any $\alpha \in \Omega$.

Definition 1.4. [17] An element $(\alpha, \beta) \in \Omega \times \Omega$ is called

(1) a coupled coincidence point of mappings $\Sigma : \Omega \times \Omega \to \Omega$ and $\phi : \Omega \to \Omega$ if $\phi(\alpha) = \Sigma(\alpha, \beta)$ and $\phi(\beta) = \Sigma(\beta, \alpha)$, and $(\phi\alpha, \phi\beta)$ is called coupled point of coincidence.
(2) a common coupled fixed point of mappings $\Sigma : \Omega \times \Omega \to \Omega$ and $\phi : \Omega \to \Omega$ if $\alpha = \phi(\alpha) = \Sigma(\alpha, \beta)$ and $\beta = \phi(\beta) = \Sigma(\beta, \alpha)$.

The notion of a partial metric space (PMS) was introduced in 1992 by Matthews [18]. Matthews proved a fixed point theorem on these spaces, analogous to Banach's fixed point theorem. Recently, many authors have focused on partial metric spaces and their topological properties (see e.g. [1, 2, 3, 8, 9, 15, 16, 17]).

The definition of a partial metric space is given by Matthews (see [18]) as follows:

Definition 1.5. Let Ω be a nonempty set and let $p : \Omega \times \Omega \to R^+$ satisfies, for all $\alpha, \beta, z \in \Omega$:

(P1) $\alpha = \beta \Longleftrightarrow p(\alpha, \alpha) = p(\beta, \beta) = p(\alpha, \beta)$;
(P2) $p(\alpha, \alpha) \leq p(\alpha, \beta)$;
(P3) $p(\alpha, \beta) = p(\beta, \alpha)$;
(P4) $p(\alpha, \beta) \leq p(\alpha, z) + p(z, \beta) - p(z, z)$.

Then the pair (Ω, p) is called a partial metric space and p is called a partial metric on Ω.

The function $d_p : \Omega \times \Omega \to R^+$ defined by

$$d_p(\alpha, \beta) = 2p(\alpha, \beta) - p(\alpha, \alpha) - p(\beta, \beta),$$

satisfies the conditions of a metric on Ω, therefore it is a (usual) metric on Ω.

Remark 1.1. (1) *If $\alpha = \beta$, $p(\alpha, \beta)$ may not be 0.*
(2) *A famous example of partial metric spaces is the pair (R^+, p), where $p(\alpha, \beta) = \max\{\alpha, \beta\}$ for all $\alpha, \beta \in R^+$.*

(3) *Each partial metric p on Ω generates a T_0 topology τ_p on Ω which has a base of open p-balls $B_p(\alpha, \varepsilon)$, where $\alpha \in \Omega$ and $\varepsilon > 0$ ($B_p(\alpha, \varepsilon) = \{\beta \in \Omega : p(\alpha, \beta) < p(\alpha, \alpha) + \varepsilon\}$).*

On a partial metric space the following concepts has been defined as follows.

Definition 1.6. [18]

(i) A sequence $\{\alpha_n\}$ in a PMS (Ω, p) converges to $\alpha \in \Omega$ iff $p(\alpha, \alpha) = \lim_n p(\alpha, \alpha_n)$.
(ii) A sequence $\{\alpha_n\}$ in a PMS (Ω, p) is called Cauchy if and only if $\lim_{n,m} p(\alpha_n, \alpha_m)$ exists (and is finite).
(iii) A PMS (Ω, p) is said to be complete if every Cauchy sequence $\{\alpha_n\}$ in Ω converges, with respect to τ_p, to a point $\alpha \in \Omega$ such that $p(\alpha, \alpha) = \lim_{n,m} p(\alpha_n, \alpha_m)$.
(iv) A mapping $f : \Omega \to \Omega$ is said to be continuous at $\alpha_0 \in \Omega$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(\alpha_0, \delta)) \subseteq B(f(\alpha_0), \varepsilon)$.

Let (Ω, p) be a partial metric space and $f : \Omega \to \Omega$ be a given mapping. Suppose that f is continuous at $\alpha_0 \in \Omega$. Then, for all sequence $\{\alpha_n\}$ in Ω, $\{\alpha_n\}$ converges to α_0 implies $f\alpha_n$ converges to $f\alpha_0$.

If $\Sigma : \Omega \times \Omega \to \Omega$ is continuous at $(\alpha_0, \beta_0) \in \Omega \times \Omega$, then for any sequences $\{\alpha_n\}$ and β_n in Ω such that $\alpha_n \to \alpha_0$ and $\beta_n \to \beta_0$ as $n \to \infty$ in (Ω, p), we have $\Sigma(\alpha_n, \beta_n) \to \Sigma(\alpha_0, \beta_0)$ as $n \to \infty$ in (Ω, p).

Some results of Shatanawi et al in [19] are the following cases.

Corollary 1.2. [19, Corollary 1] *Let $(\Omega, \preceq)$ be a partially ordered set and p be a partial metric on Ω such that (Ω, p) is a complete partial metric space. Let $\Sigma : \Omega \times \Omega \to \Omega$ be a continuous mapping having the mixed monotone property on Ω. Assume that for $\alpha, \beta, u, v \in \Omega$ with $\alpha \preceq u$ and $\beta \succeq v$, we have*

$$p(\Sigma(\alpha, \beta), \Sigma(u, v)) \leq \frac{k}{2}[p(\alpha, u) + p(\beta, v)],$$

where $k \in [0, 1)$. If there exists $(\alpha_0, \beta_0) \in \Omega \times \Omega$ such that $\alpha_0 \preceq \Sigma(\alpha_0, \beta_0)$ and $\beta_0 \succeq \Sigma(\beta_0, \alpha_0)$, then Σ has a coupled fixed point.

Corollary 1.3. [19, Corollary 2] *Let $(\Omega, \preceq)$ be a partially ordered set and p be a partial metric on Ω such that (Ω, p) is a complete partial metric space. Let $\Sigma : \Omega \times \Omega \to \Omega$ be a mapping having the mixed monotone property on Ω. Assume that for $\alpha, \beta, \upsilon, \nu \in \Omega$ with $\alpha \preceq \upsilon$ and $\beta \succeq \upsilon$, we have*

$$p(\Sigma(\alpha, \beta), \Sigma(\upsilon, \nu)) \leq \frac{k}{2}[p(\alpha, \upsilon) + p(\beta, \upsilon)],$$

where $k \in [0, 1)$. Also, suppose that Ω has the following properties:

(i) *if $\{\alpha_n\}$ is a nondecreasing sequence and $\alpha \in \Omega$ with $\lim_{n\to\infty} p(\alpha_n, \alpha) = p(\alpha, \alpha) = 0$, then $\alpha_n \preceq \alpha$ for all n,*

(ii) *if* $\{\alpha_n\}$ *is a nonincreasing sequence and* $\alpha \in \Omega$ *with* $\lim_{n\to\infty} p(\alpha_n, \alpha) = p(\alpha, \alpha) = 0$, *then* $\alpha \preceq \alpha_n$ *for all* n.

If there exists $(\alpha_0, \beta_0) \in \Omega \times \Omega$ *such that* $\alpha_0 \preceq \Sigma(\alpha_0, \beta_0)$ *and* $\beta_0 \succeq \Sigma(\beta_0, \alpha_0)$, *then* Σ *has a coupled fixed point.*

In this paper, we establish some coupled coincidence point results of nonlinear contraction mappings in the framework of ordered partial metric spaces. Our results extend and generalize the results of Shatanawi et al. [**19**].

2. Main results

We recall three easy lemmas which have a essential role in the proof of the main results. This results can be derived easily (see e.g. [**2, 18**]).

Lemma 2.1. *(1) A sequence* $\{\alpha_n\}$ *is a Cauchy sequence in the PMS* (Ω, p) *iff it is a Cauchy sequence in the metric space* (Ω, d_p).

(2) A PMS (Ω, p) *is complete iff the metric space* (Ω, d_p) *is complete. Moreover,*

$$\lim_n d_p(\alpha, \alpha_n) = 0 \Longleftrightarrow p(\alpha, \alpha) = \lim_n p(\alpha, \alpha_n) = \lim_{n,m} p(\alpha_n, \alpha_m).$$

Lemma 2.2. [**1**] *Assume that* $\alpha_n \to z$ *as* $n \to \infty$ *in a PMS* (Ω, p) *such that* $p(z, z) = 0$. *Then* $\lim_n p(\alpha_n, \beta) = p(z, \beta)$ *for every* $\beta \in \Omega$.

Lemma 2.3. [**1, 15**] *Let* (Ω, p) *be a PMS. Then,*
(A) If $p(\alpha, \beta) = 0$ *then* $\alpha = \beta$.
(B) If $\alpha \neq \beta$, *then* $p(\alpha, \beta) > 0$.

We define a notion of compatibility in the following:

Definition 2.1. The mappings Σ and ϕ where $\Sigma : \Omega \times \Omega \to \Omega$ and $\phi : \Omega \to \Omega$, are said to be partial-compatible if

(1)
$$\lim_{n\to\infty} p(\phi(\Sigma(\alpha_n, \beta_n)), \Sigma(\phi(\alpha_n), \phi(\beta_n))) = 0,$$
and
$$\lim_{n\to\infty} p(\phi(\Sigma(\beta_n, \alpha_n)), \Sigma(\phi(\beta_n), (\phi\alpha_n))) = 0,$$
whenever $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in Ω such that $\Sigma(\alpha_n, \beta_n) \to \alpha$, $\phi(\alpha_n) \to \alpha$, $\Sigma(\beta_n, \alpha_n) \to \beta$ and $\phi(\beta_n) \to \beta$, for some $\alpha, \beta \in \Omega$;

(2) $p(\alpha, \alpha) = 0$ implies that $p(\phi\alpha, \phi\alpha) = 0$.

Note that the above definition extends and generalizes the notion of compatibility introduced by Choudhury and Kundu [**7**].

Our main result is the following.

Theorem 2.4. *Let $(\Omega, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on Ω such that (Ω, p) be a complete partial metric space. Let $\Sigma : \Omega \times \Omega \to \Omega$ and $\alpha : \Omega \to \Omega$ be two mappings such that*

$$\begin{aligned} &p(\Sigma(\alpha,\beta),\Sigma(\upsilon,\nu)) \\ &\leq \alpha_1 p(\phi\alpha,\phi u) + \alpha_2 p(\phi y,\phi v) + \alpha_3 p(\Sigma(\alpha,\beta),\phi\alpha) + \alpha_4 p(\Sigma(\beta,\alpha),\phi\beta) \\ &\quad +\alpha_5 p(\Sigma(\alpha,\beta),\phi u) + \alpha_6 p(\Sigma(\beta,\alpha),\phi v) + \alpha_7 p(\Sigma(\upsilon,\nu),\phi\alpha) \\ &\quad +\alpha_8 p(\Sigma(\upsilon,u),\phi\beta) + \alpha_9 p(\Sigma(\upsilon,\nu),\phi u) + \alpha_{10} p(\Sigma(\upsilon,u),\phi v)), \end{aligned} \tag{1}$$

for every pairs $(\alpha,\beta),(\upsilon,\nu) \in \Omega \times \Omega$ such that $\phi\alpha \preceq \phi u$ and $\phi\beta \succeq \phi v$, where

(1) $\alpha_7 + \alpha_8 + \sum_{i=1}^{10} \alpha_i < 1$, *if* $\alpha_5 - \alpha_7 < 0$ *and* $\alpha_6 - \alpha_8 < 0$;
(2) $\alpha_8 + \sum_{i=1}^{10} \alpha_i < 1$, *if* $\alpha_5 - \alpha_7 \geq 0$ *and* $\alpha_6 - \alpha_8 < 0$;
(3) $\alpha_7 + \sum_{i=1}^{10} \alpha_i < 1$, *if* $\alpha_5 - \alpha_7 < 0$ *and* $\alpha_6 - \alpha_8 \geq 0$;
(4) $\sum_{i=1}^{10} \alpha_i < 1$, *if* $\alpha_5 - \alpha_7 \geq 0$ *and* $\alpha_6 - \alpha_8 \geq 0$.

Suppose that

(1) $\Sigma(\Omega \times \Omega) \subseteq \phi(\Omega)$;
(2) Σ *has the mixed g-monotone property;*
(3) ϕ *is continuous and monotone increasing and Σ and g be partial-compatible mappings.*

Also, suppose

(a) Σ *is continuous, or,*
(b) Ω *has the following properties:*
(i) *if $\{\alpha_n\}$ is a nondecreasing sequence and $\alpha \in \Omega$ with $\lim_{n\to\infty} p(\alpha_n,\alpha) = p(\alpha,\alpha) = 0$, then $\alpha_n \preceq \alpha$ for all n,*
(ii) *if $\{\alpha_n\}$ is a nonincreasing sequence and $\alpha \in \Omega$ with $\lim_{n\to\infty} p(\alpha_n,\alpha) = p(\alpha,\alpha) = 0$, then $\alpha \preceq \alpha_n$ for all n.*

If there exist $\alpha_0, \beta_0 \in \Omega$ such that $\phi\alpha_0 \preceq \Sigma(\alpha_0,\beta_0)$ and $\Sigma(\beta_0,\alpha_0) \preceq \phi\beta_0$, then Σ and ϕ have a coupled coincidence point in Ω.

Proof. Let $\alpha_0, \beta_0 \in \Omega$ be such that $\phi\alpha_0 \preceq \Sigma(\alpha_0,\beta_0)$ and $\phi\beta_0 \succeq \Sigma(\beta_0,\alpha_0)$. Since $\Sigma(\Omega \times \Omega) \subseteq g(\Omega)$, we can define $\alpha_1, \beta_1 \in \Omega$ such that $\phi\alpha_1 = \Sigma(\alpha_0,\beta_0)$ and $\phi\beta_1 = \Sigma(\beta_0,\alpha_0)$, then $\phi\alpha_0 \preceq \Sigma(\alpha_0,\beta_0) = \phi\alpha_1$ and $\phi\beta_0 \succeq \Sigma(\beta_0,\alpha_0) = \phi\beta_1$. Since Σ has the mixed g-monotone property, we have $\Sigma(\alpha_0,\beta_0) \preceq \Sigma(\alpha_1,\beta_0) \preceq \Sigma(\alpha_1,\beta_1)$ and $\Sigma(\beta_0,\alpha_0) \succeq \Sigma(\beta_1,\alpha_0) \succeq \Sigma(\beta_1,\alpha_1)$. In this way we construct the sequences z_n and t_n inductively as $z_n = \phi\alpha_n = \Sigma(\alpha_{n-1}, y_{n-1})$, and $t_n = \phi\beta_n = \Sigma(y_{n-1},\alpha_{n-1})$, for all $n \geq 0$.

We can easily show that for all $n \geq 0$, $z_{n-1} \preceq z_n$, and $t_{n-1} \succeq t_n$. We complete the proof in three steps.

Step I. Set

$$\delta_n = p(z_{n-1}, z_n) + p(t_{n-1}, t_n),$$

we shall show that

$$\lim_{n\to\infty} \delta_n = 0.$$

Since $\phi\alpha_{n-1} \preceq \phi\alpha_n$ and $\phi\beta_{n-1} \succeq \phi\alpha_n$, using (1), we obtain

$$\begin{aligned}
&p(z_n, z_{n+1}) = p(\Sigma(\alpha_{n-1}, y_{n-1}), \Sigma(\alpha_n, y_n))\\
&\leq \alpha_1 p(\phi\alpha_{n-1}, \phi\alpha_n) + \alpha_2 p(\phi\beta_{n-1}, gy_n) + \alpha_3 p(\Sigma(\alpha_{n-1}, y_{n-1}), \phi\alpha_{n-1})\\
&\quad +\alpha_4 p(\Sigma(y_{n-1}, \alpha_{n-1}), gy_{n-1}) + \alpha_5 p(\Sigma(\alpha_{n-1}, \beta_{n-1}), \phi\alpha_n)\\
&\quad +\alpha_6 p(\Sigma(\beta_{n-1}, \alpha_{n-1}), \phi\beta_n) + \alpha_7 p(\Sigma(\alpha_n, \beta_n), \phi\alpha_{n-1})\\
&\quad +\alpha_8 p(\Sigma(\beta_n, \alpha_n), \phi\beta_{n-1}) + \alpha_9 p(\Sigma(\alpha_n, \beta_n), \phi\alpha_n) + \alpha_{10} p(\Sigma(\beta_n, \alpha_n), \phi\beta_n)\\
&= \alpha_1 p(z_{n-1}, z_n) + \alpha_2 p(t_n, t_{n-1}) + \alpha_3 p(z_n, z_{n-1}) + \alpha_4 p(t_n, t_{n-1}) + \alpha_5 p(z_n, z_n)\\
&\quad +\alpha_6 p(t_n, t_n) + \alpha_7 p(z_{n+1}, z_{n-1}) + \alpha_8 p(t_{n+1}, t_{n-1}) + \alpha_9 p(z_{n+1}, z_n) + \alpha_{10} p(t_{n+1}, t_n)\\
&\leq \alpha_1 p(z_{n-1}, z_n) + \alpha_2 p(t_{n-1}, t_n) + \alpha_3 p(z_n, z_{n-1}) + \alpha_4 p(t_n, t_{n-1})\\
&\quad +(\alpha_5 - \alpha_7) p(z_n, z_n) + (\alpha_6 - \alpha_8) p(t_n, t_n)\\
&\quad +\alpha_7 [p(z_{n+1}, z_n) + p(z_n, z_{n-1})] + \alpha_8 [p(t_{n+1}, t_n) + p(t_n, t_{n-1})]\\
&\quad +\alpha_9 p(z_{n+1}, z_n) + \alpha_{10} p(t_{n+1}, t_n).
\end{aligned} \tag{2}$$

In a similar way, as $\phi\beta_{n-1} \succeq \phi\beta_n$ and $\phi\alpha_{n-1} \preceq \phi\alpha_n$, we have

$$\begin{aligned}
&p(t_n, t_{n+1}) = p(\Sigma(y_{n-1}, \alpha_{n-1}), \Sigma(y_n, \alpha_n))\\
&\leq \alpha_1 p(t_{n-1}, t_n) + \alpha_2 p(z_{n-1}, z_n) + \alpha_3 p(t_n, t_{n-1}) + \alpha_4 p(z_n, z_{n-1})\\
&\quad +(\alpha_5 - \alpha_7) p(t_n, t_n) + (\alpha_6 - \alpha_8) p(z_n, z_n)\\
&\quad +\alpha_7 [p(t_{n+1}, t_n) + p(t_n, t_{n-1})] + \alpha_8 [p(z_{n+1}, z_n) + p(z_n, z_{n-1})]\\
&\quad +\alpha_9 p(t_{n+1}, t_n) + \alpha_{10} p(z_{n+1}, z_n).
\end{aligned} \tag{3}$$

We have four cases:

Case 1. $\alpha_5 - \alpha_7 < 0$ and $\alpha_6 - \alpha_8 < 0$. As $p(z_n, z_n) \leq p(z_{n+1}, z_n)$ and $p(t_n, t_n) \leq p(t_{n+1}, t_n)$, removing the terms $\alpha_7 p(z_n, z_n)$, $\alpha_8 p(z_n, z_n)$, $\alpha_7 p(t_n, t_n)$ and $\alpha_8 p(t_n, t_n)$ and adding (2) and (3), we get

$$\begin{aligned}
\delta_{n+1} &\leq [\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8]\delta_n\\
&\quad +[\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}]\delta_{n+1}.
\end{aligned} \tag{4}$$

Case 2. $\alpha_5 - \alpha_7 \geq 0$ and $\alpha_6 - \alpha_8 < 0$. Adding, (2) and (3), by a careful computation, we infer

$$\begin{aligned}
\delta_{n+1} &\leq [\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8]\delta_n\\
&\quad +[\alpha_5 + \alpha_6 + \alpha_8 + \alpha_9 + \alpha_{10}]\delta_{n+1}.
\end{aligned} \tag{5}$$

Case 3. $\alpha_5 - \alpha_7 < 0$ and $\alpha_6 - \alpha_8 \geq 0$. Similarly, we have

$$\begin{aligned}
\delta_{n+1} &\leq [\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8]\delta_n\\
&\quad +[\alpha_5 + \alpha_7 + \alpha_6 + \alpha_9 + \alpha_{10}]\delta_{n+1}.
\end{aligned} \tag{6}$$

Case 4. $\alpha_5 - \alpha_7 \geq 0$ and $\alpha_6 - \alpha_8 \geq 0$. Again, we can obtain that

$$\begin{aligned}
\delta_{n+1} &\leq [\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8]\delta_n\\
&\quad +[\alpha_5 + \alpha_6 + \alpha_9 + \alpha_{10}]\delta_{n+1}.
\end{aligned} \tag{7}$$

Repeating the above process, we have

$$\delta_{n+1} \leq \alpha\delta_n \leq \alpha^2\delta_{n-1} \leq \cdots \leq \alpha^{n+1}\delta_0, \tag{8}$$

where $\alpha \in [0,1)$. Hence,

$$\lim_{n\to\infty} \delta_n = 0. \tag{9}$$

Step II. $\{z_n\}$ and $\{t_n\}$ are Cauchy. Now, we claim that $\lim\limits_{n,m\to\infty} p(z_n, z_m) + p(t_n, t_m) = 0$.

Suppose to the contrary, then there exists $\varepsilon > 0$ for which we can find subsequences $\{z_{m(k)}\}$ and $\{z_{n(k)}\}$ of $\{z_n\}$ and $\{t_{m(k)}\}$ and $\{t_{n(k)}\}$ of $\{t_n\}$ such that $n(k) > m(k) > k$ and $p(z_{m(k)}, z_{n(k)}) + p(t_{m(k)}, t_{n(k)}) \geq \varepsilon$, where $n(k)$ is the smallest index with this property, i.e.,

$$p(z_{m(k)}, z_{n(k)-1}) + p(t_{m(k)}, t_{n(k)-1}) < \varepsilon. \tag{10}$$

From triangle inequality,

$$\begin{aligned}\varepsilon \leq\ & p(z_{m(k)}, z_{n(k)}) + p(t_{m(k)}, t_{n(k)}) \\ & \leq p(z_{m(k)}, z_{n(k)-1}) + p(z_{n(k)-1}, z_{n(k)}) \\ & \quad + p(t_{m(k)}, t_{n(k)-1}) + p(t_{n(k)-1}, t_{n(k)}) \\ & < \varepsilon + p(z_{n(k)-1}, z_{n(k)}) + p(t_{n(k)-1}, t_{n(k)}) = \varepsilon + \delta_{n(k)}.\end{aligned} \tag{11}$$

If $k \to \infty$, as $\lim\limits_{n\to\infty} \delta_n = 0$, we conclude that

$$\lim_{k\to\infty} p(z_{m(k)}, z_{n(k)}) + p(t_{m(k)}, t_{n(k)}) = \varepsilon. \tag{12}$$

Since,

$$\begin{aligned}& p(z_{m(k)+1}, z_{n(k)}) + p(t_{m(k)+1}, t_{n(k)}) \\ & \leq p(z_{m(k)+1}, z_{m(k)}) + p(t_{m(k)+1}, t_{m(k)}) + p(z_{m(k)}, z_{n(k)}) + p(t_{m(k)}, t_{n(k)}) \\ & = \delta_{m(k)+1} + [p(z_{m(k)}, z_{n(k)}) + p(t_{m(k)}, t_{n(k)})],\end{aligned} \tag{13}$$

and

$$\begin{aligned}& p(z_{m(k)}, z_{n(k)}) + p(t_{m(k)}, t_{n(k)}) \\ & \leq p(z_{m(k)}, z_{m(k)+1}) + p(t_{m(k)}, t_{m(k)+1}) + p(z_{m(k)+1}, z_{n(k)}) + p(t_{m(k)+1}, t_{n(k)}) \\ & = \delta_{m(k)+1} + [p(z_{m(k)+1}, z_{n(k)}) + p(t_{m(k)+1}, t_{n(k)})],\end{aligned} \tag{14}$$

we have

$$\lim_{k\to\infty} p(z_{m(k)+1}, z_{n(k)}) + p(t_{m(k)+1}, t_{n(k)}) = \varepsilon. \tag{15}$$

In a similar way, we get

$$\lim_{k\to\infty} p(z_{n(k)+1}, z_{m(k)}) + p(t_{n(k)+1}, t_{m(k)}) = \varepsilon. \tag{16}$$

Since,

$$\begin{aligned}& p(z_{m(k)+1}, z_{n(k)+1}) + p(t_{m(k)+1}, t_{n(k)+1}) \\ & \leq p(z_{m(k)+1}, z_{m(k)}) + p(t_{m(k)+1}, t_{m(k)}) + p(z_{m(k)}, z_{n(k)+1}) + p(t_{m(k)}, t_{n(k)+1}) \\ & = \delta_{m(k)+1} + [p(z_{m(k)}, z_{n(k)+1}) + p(t_{m(k)}, t_{n(k)+1})],\end{aligned} \tag{17}$$

and

$$\begin{aligned} &p(z_{m(k)}, z_{n(k)+1}) + p(t_{m(k)}, t_{n(k)+1}) \\ &\leq p(z_{m(k)}, z_{m(k)+1}) + p(t_{m(k)}, t_{m(k)+1}) + p(z_{m(k)+1}, z_{n(k)+1}) + p(t_{m(k)+1}, t_{n(k)+1}) \\ &= \delta_{m(k)+1} + [p(z_{m(k)+1}, z_{n(k)+1}) + p(t_{m(k)+1}, t_{n(k)+1})], \end{aligned} \tag{18}$$

we have

$$\lim_{k\to\infty} p(z_{m(k)+1}, z_{n(k)+1}) + p(t_{m(k)+1}, t_{n(k)+1}) = \varepsilon. \tag{19}$$

As $\phi\alpha_{m(k)} \preceq \phi\alpha_{n(k)}$ and $\phi\beta_{m(k)} \succeq \phi\beta_{n(k)}$, putting $\alpha = \alpha_{m(k)}, \beta = y_{m(k)}, u = \alpha_{n(k)}$ and $v = y_{n(k)}$ in (1), for all $k \geq 0$, we get

$$\begin{aligned} &p(z_{m(k)+1}, z_{n(k)+1}) = p(\Sigma(\alpha_{m(k)}, y_{m(k)}), \Sigma(\alpha_{n(k)}, y_{n(k)})) \\ &\leq \alpha_1 p(\phi\alpha_{m(k)}, \phi\alpha_{n(k)}) + \alpha_2 p(\phi\beta_{m(k)}, \phi\beta_{n(k)}) \\ &\quad +\alpha_3 p(\Sigma(\alpha_{m(k)}, y_{m(k)}), \phi\alpha_{m(k)}) + \alpha_4 p(\Sigma(y_{m(k)}, \alpha_{m(k)}), \phi\beta_{m(k)}) \\ &\quad +\alpha_5 p(\Sigma(\alpha_{m(k)}, \beta_{m(k)}), \phi\alpha_{n(k)}) + \alpha_6 p(\Sigma(\beta_{m(k)}, \alpha_{m(k)}), \phi\beta_{n(k)}) \\ &\quad +\alpha_7 [p(\Sigma(\alpha_{n(k)}, \beta_{n(k)}), \phi\alpha_{m(k)}) + \alpha_8 p(\Sigma(\beta_{n(k)}, \alpha_{n(k)}), \phi\beta_{m(k)}) \\ &\quad +\alpha_9 p(\Sigma(\alpha_{n(k)}, \beta_{n(k)}), \phi\alpha_{n(k)}) + \alpha_{10} p(\Sigma(\beta_{n(k)}, \alpha_{n(k)}), \phi\beta_{n(k)}) \\ &= \alpha_1 p(z_{m(k)}, z_{n(k)}) + \alpha_2 p(t_{m(k)}, t_{n(k)}) \\ &\quad +\alpha_3 p(z_{m(k)+1}, z_{m(k)}) + \alpha_4 p(t_{m(k)+1}, t_{m(k)}) \\ &\quad +\alpha_5 p(z_{m(k)+1}, z_{n(k)}) + \alpha_6 p(t_{m(k)+1}, t_{n(k)}) \\ &\quad +\alpha_7 [p(z_{n(k)+1}, z_{m(k)}) + \alpha_8 p(t_{n(k)+1}, t_{m(k)}) \\ &\quad +\alpha_9 p(z_{n(k)+1}, z_{n(k)}) + \alpha_{10} p(t_{n(k)+1}, t_{n(k)}). \end{aligned} \tag{20}$$

Also,

$$\begin{aligned} &p(t_{m(k)+1}, t_{n(k)+1}) = p(\Sigma(y_{m(k)}, \alpha_{m(k)}), \Sigma(y_{n(k)}, \alpha_{n(k)})) \\ &\leq \alpha_1 p(t_{m(k)}, t_{n(k)}) + \alpha_2 p(z_{m(k)}, z_{n(k)}) \\ &\quad +\alpha_3 p(t_{m(k)+1}, t_{m(k)}) + \alpha_4 p(z_{m(k)+1}, z_{m(k)}) \\ &\quad +\alpha_5 p(t_{m(k)+1}, t_{n(k)}) + \alpha_6 p(z_{m(k)+1}, z_{n(k)}) \\ &\quad +\alpha_7 [p(t_{n(k)+1}, t_{m(k)}) + \alpha_8 p(z_{n(k)+1}, z_{m(k)}) \\ &\quad +\alpha_9 p(t_{n(k)+1}, t_{n(k)}) + \alpha_{10} p(z_{n(k)+1}, z_{n(k)}). \end{aligned} \tag{21}$$

Adding (20) and (21), we obtain

$$\begin{aligned} &p(z_{m(k)+1}, z_{n(k)+1}) + p(t_{m(k)+1}, t_{n(k)+1}) \\ &\leq (\alpha_1 + \alpha_2)[p(z_{m(k)}, z_{n(k)}) + p(t_{m(k)}, t_{n(k)})] \\ &\quad +(\alpha_3 + \alpha_4) p(z_{m(k)+1}, z_{m(k)}) + p(t_{m(k)+1}, t_{m(k)})] \\ &\quad +(\alpha_5 + \alpha_6)[p(z_{m(k)+1}, z_{n(k)}) + p(t_{m(k)+1}, t_{n(k)})] \\ &\quad +(\alpha_7 + \alpha_8)[p(z_{m(k)}, z_{n(k)+1}) + p(t_{m(k)}, t_{n(k)+1})] \\ &\quad +(\alpha_9 + \alpha_{10}) p(z_{n(k)+1}, z_{n(k)}) + p(t_{n(k)+1}, t_{n(k)})]. \end{aligned} \tag{22}$$

Now, if $k \to \infty$ in (22), from (9), (12), (15), (16) and (19), we have

$$\varepsilon \leq (\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)\varepsilon < \varepsilon, \tag{23}$$

which is a contradiction.

This proves that $\{z_n\}$ and $\{t_n\}$ are Cauchy sequences in (Ω, p) and hence $\{z_n\}$ and $\{t_n\}$ are Cauchy sequences in the metric space (Ω, d_p). From Lemma 2.1, (Ω, d_p) is complete, so $\{z_n\}$ and $\{t_n\}$ converges to some $z, t \in \Omega$, respectively, that is $\lim_{n\to\infty} d_p(z_n, z) = 0$ and $\lim_{n\to\infty} d_p(t_n, t) = 0$. Therefore, from Lemma 2.1, we have

$$p(z,z) = \lim_{n\to\infty} p(z_n, z) = \lim_{n,m\to\infty} p(z_n, z_m) = 0, \tag{24}$$

and

$$p(t,t) = \lim_{n\to\infty} p(t_n, z) = \lim_{n,m\to\infty} p(t_n, t_m) = 0. \tag{25}$$

Step III. We show that Σ and g have a coupled coincidence point. From the above step,

$$\lim_{n\to\infty} p(\Sigma(\alpha_n, \beta_n), z) = \lim_{n\to\infty} p(\phi\alpha_n, z) = 0, \tag{26}$$

and

$$\lim_{n\to\infty} p(\Sigma(\beta_n, \alpha_n), t) = \lim_{n\to\infty} p(\phi\beta_n, t) = 0. \tag{27}$$

Since Σ and g are partial-compatible, from (24), (25), (26) and (27), we deduce

$$\lim_{n\to\infty} p(\phi(\Sigma(\alpha_n, \beta_n)), \Sigma(\phi\alpha_n, \phi\beta_n)) = \lim_{n\to\infty} p(\phi(\Sigma(\beta_n, \alpha_n)), \Sigma(\phi\beta_n, \phi\alpha_n)) = 0, \tag{28}$$

and

$$p(\phi z, \phi z) = p(\phi t, \phi t) = 0. \tag{29}$$

Next, we prove that $\phi z = \Sigma(z,t)$ and $\phi t = \Sigma(t,z)$. Suppose that (a) is satisfied. For all $n \geq 0$, we have

$$p(\phi z, \Sigma(\phi\alpha_n, \phi\beta_n)) \leq p(\phi z, \phi(\Sigma(\alpha_n, \beta_n))) + p(\phi(\Sigma(\alpha_n, \beta_n)), \Sigma(\phi\alpha_n, \phi\beta_n)).$$

Taking the limit as $n \to \infty$, (26), (27), (28), (29), using Lemma 2.2 and the fact that Σ and ϕ are continuous, we have $p(\phi z, \Sigma(z,t)) = 0$. Similarly, we have $p(\phi t, \Sigma(t,z)) = 0$. Combining the above two results we get $\phi z = \Sigma(z,t)$ and $\phi t = \Sigma(t,z)$.

Next, we suppose that (b) holds. We know that $\{\phi\alpha_n\}$ is nondecreasing sequence, $\phi\alpha_n \to z$ and $\{\phi\beta_n\}$ is nonincreasing sequence, $\phi\beta_n \to t$ as $n \to \infty$. Then by (i) and (ii), we have for all $n \geq 0$, $\phi\alpha_n \preceq z$ and $\phi\beta_n \succeq t$.

Since ϕ is continuous, by (26) and (27), we get

$$\lim_{n\to\infty} p(\phi(\phi\alpha_n), gz) = \lim_{n\to\infty} p(\phi(\phi\beta_n), gt) = 0. \tag{30}$$

Now,

$$p(\phi z, \Sigma(z,t)) \leq p(\phi z, \phi(\phi\alpha_{n+1})) + p(\phi\phi(\alpha_{n+1}), \Sigma(z,t)). \tag{31}$$

Taking limit as $n \to \infty$ in the above inequality, as $\phi\alpha_{n+1} = \Sigma(\alpha_n, \beta_n)$ and using triangle inequality, (28) and (30), we infer

$$\begin{aligned} &p(\phi z, \Sigma(z,t)) \\ &\le \lim_{n\to\infty} p(gz, \phi(\phi\alpha_{n+1})) + \lim_{n\to\infty} p(\phi(\Sigma(\alpha_n,\beta_n)), \Sigma(\phi\alpha_n, \phi\beta_n)) \\ &\quad + \lim_{n\to\infty} p(\Sigma(g\alpha_n, \phi\beta_n), \Sigma(z,t)) \\ &\le \lim_{n\to\infty} p(\Sigma(\phi\alpha_n, \phi\beta_n), \Sigma(z,t)). \end{aligned}$$

On the other hand, since ϕ is monotone increasing, by (i), (ii) and (30), we have $\phi(\phi\alpha_n) \preceq gz$ and $\phi(\phi\beta_n) \succeq \phi t$. So, for all $n \ge 0$, from (1)

$$\begin{aligned} &p(\Sigma(\phi\alpha_n, \phi y_n), \Sigma(z,t)) \\ &\le \alpha_1 p(\phi\phi\alpha_n, \phi z) + \alpha_2 p(\phi\phi\beta_n, gt) + \alpha_3 p(\Sigma(\phi\alpha_n, \phi\beta_n), \phi\phi\alpha_n) \\ &\quad + \alpha_4 p(\Sigma(\phi\beta_n, \phi\alpha_n), \phi\phi\beta_n) + \alpha_5 p(\Sigma(\phi\alpha_n, \phi\beta_n), \phi z) \\ &\quad + \alpha_6 p(\Sigma(\phi\beta_n, \phi\alpha_n), \phi t) + \alpha_7 p(\Sigma(z,t), \phi\phi\alpha_n) \\ &\quad + \alpha_8 p(\Sigma(t,z), ggy_n) + \alpha_9 p(\Sigma(z,t), \phi z) + \alpha_{10} p(\Sigma(t,z), \phi t). \end{aligned} \tag{32}$$

In the above inequality, if $n \to \infty$

$$\begin{aligned} &\lim_{n\to\infty} p(\Sigma(\phi\alpha_n, \phi\beta_n), \Sigma(z,t)) \\ &\le (\alpha_7 + \alpha_9) p(\Sigma(z,t), \phi z) + (\alpha_8 + \alpha_{10}) p(\Sigma(t,z), \phi t). \end{aligned} \tag{33}$$

Analogously,

$$p(\phi t, \Sigma(t,z)) \le p(\phi t, \phi(\phi\beta_{n+1})) + p(\phi\phi(\beta_{n+1}), \Sigma(t,z)). \tag{34}$$

Taking limit as $n \to \infty$ in the above inequality, since $\phi\beta_{n+1} = \Sigma(\beta_n, \alpha_n)$ and using triangle inequality, (28) and (30), we have

$$\begin{aligned} &p(\phi t, \Sigma(t,z)) \\ &\le \lim_{n\to\infty} p(gt, \phi(\phi\beta_{n+1})) + \lim_{n\to\infty} p(\phi(\Sigma(\beta_n, \alpha_n), \Sigma(\phi\beta_n, \phi\alpha_n)) \\ &\quad + \lim_{n\to\infty} p(\Sigma(\phi\beta_n, g\alpha_n)), \Sigma(t,z)) \\ &\le \lim_{n\to\infty} p(\Sigma(\phi\beta_n, \phi\alpha_n), \Sigma(t,z)). \end{aligned}$$

Similar to the above argument, we get

$$\begin{aligned} &\lim_{n\to\infty} p(\Sigma(\phi\beta_n, \phi\alpha_n), \Sigma(t,z)) \\ &\le (\alpha_7 + \alpha_9) p(\Sigma(t,z), \phi t) + (\alpha_8 + \alpha_{10}) p(\Sigma(z,t), \phi z). \end{aligned} \tag{35}$$

Adding, (33) and (35) and using (31) and (34), we obtain

$$\begin{aligned} &p(\phi z, \Sigma(z,t)) + p(\phi t, \Sigma(t,z)) \\ &\le (\alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10})[p(\phi z, \Sigma(z,t)) + p(\phi t, \Sigma(t,z))]. \end{aligned} \tag{36}$$

Therefore, $p(\phi z, \Sigma(z,t)) + p(\phi t, \Sigma(t,z)) = 0$, that is, $\Sigma(z,t) = \phi z$ and $\Sigma(t,z) = \phi t$. □

Many results can be deduced from the above theorem as follows.

Corollary 2.5. *Let $(\Omega, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on Ω such that (Ω, p) be a complete partial metric space. Let $\Sigma : \Omega \times \Omega \to \Omega$ and $g : \Omega \to \Omega$ be two mappings such that*

$$p(\Sigma(\alpha, \beta), \Sigma(\upsilon, \nu)) \leq \alpha_1 p(\phi\alpha, \phi u) + \alpha_2 p(\phi\beta, \phi v), \tag{37}$$

for every pairs $(\alpha, \beta), (\upsilon, \nu) \in \Omega \times \Omega$ such that $\phi\alpha \preceq \phi u$ and $\phi\beta \succeq \phi v$, where $\alpha_1 + \alpha_2 < 1$. Suppose that

(1) *$\Sigma(\Omega \times \Omega) \subseteq \phi(\Omega)$;*
(2) *Σ has the mixed g-monotone property;*
(3) *ϕ is continuous and monotone increasing and Σ and g be partial-compatible mappings.*

Also, suppose that

(4) *Σ is continuous, or,*
(5) *Ω has the following properties:*
(i) *if $\{\alpha_n\}$ is a nondecreasing sequence and $\alpha \in \Omega$ with $\lim_{n\to\infty} p(\alpha_n, \alpha) = p(\alpha, \alpha) = 0$, then $\alpha_n \preceq \alpha$ for all n,*
(ii) *if $\{\alpha_n\}$ is a nonincreasing sequence and $\alpha \in \Omega$ with $\lim_{n\to\infty} p(\alpha_n, \alpha) = p(\alpha, \alpha) = 0$, then $\alpha \preceq \alpha_n$ for all n.*

If there exist $\alpha_0, \beta_0 \in \Omega$ such that $\phi\alpha_0 \preceq \Sigma(\alpha_0, \beta_0)$ and $\Sigma(\beta_0, \alpha_0) \preceq \phi\beta_0$, then Σ and ϕ have a coupled coincidence point in Ω.

Remark 2.6. *If in the above Corollary, we assume $\phi(\alpha) = \alpha$ for all $\alpha \in \Omega$ and $\alpha_1 = \alpha_2$, then we obtain the results of Shatanawi et al. in [19] which are noted here in Corollaries 1.2 and 1.3.*

Corollary 2.7. *Let $(\Omega, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on Ω such that (Ω, p) be a complete partial metric space. Let $\Sigma : \Omega \times \Omega \to \Omega$ and $g : \Omega \to \Omega$ be two mappings such that*

$$\begin{aligned} p(\Sigma(\alpha, \beta), \Sigma(\upsilon, \nu)) \ &\leq \alpha_1 p(\Sigma(\alpha, \beta), \phi\alpha) + \alpha_2 p(\Sigma(\beta, \alpha), \phi\beta) \\ &\quad + \alpha_3 p(\Sigma(\upsilon, \nu), \phi u) + \alpha_4 p(\Sigma(v, u), \phi v)), \end{aligned} \tag{38}$$

for every pairs $(\alpha, \beta), (\upsilon, \nu) \in \Omega \times \Omega$ such that $\phi\alpha \preceq \phi u$ and $\phi\beta \succeq \phi v$, where $\sum_{i=1}^{4} \alpha_i < 1$.
Suppose that

(1) *$\Sigma(\Omega \times \Omega) \subseteq \phi(\Omega)$;*
(2) *Σ has the mixed g-monotone property;*
(3) *ϕ is continuous and monotone increasing and Σ and g be partial-compatible mappings.*

Also, suppose that

(a) Σ *is continuous, or,*
(b) Ω *has the following properties:*
(i) *if* $\{\alpha_n\}$ *is a nondecreasing sequence and* $\alpha \in \Omega$ *with* $\lim_{n\to\infty} p(\alpha_n, \alpha) = p(\alpha, \alpha) = 0$*, then* $\alpha_n \preceq \alpha$ *for all* n*,*
(ii) *if* $\{\alpha_n\}$ *is a nonincreasing sequence and* $\alpha \in \Omega$ *with* $\lim_{n\to\infty} p(\alpha_n, \alpha) = p(\alpha, \alpha) = 0$*, then* $\alpha \preceq \alpha_n$*, for all* n*.*

If there exist $\alpha_0, \beta_0 \in \Omega$ *such that* $\phi\alpha_0 \preceq \Sigma(\alpha_0, \beta_0)$ *and* $\Sigma(\beta_0, \alpha_0) \preceq \phi\beta_0$*, then* Σ *and* ϕ *have a coupled coincidence point in* Ω*.*

Corollary 2.8. *Let* $(\Omega, \preceq)$ *be a partially ordered set and suppose that there is a partial metric* p *on* Ω *such that* (Ω, p) *be a complete partial metric space. Let* $\Sigma : \Omega \times \Omega \to \Omega$ *and* $g : \Omega \to \Omega$ *be two mappings such that*

$$\begin{aligned} p(\Sigma(\alpha,\beta),\Sigma(\upsilon,\nu)) \ &\leq \alpha_1 p(\Sigma(\alpha,\beta),\phi u) + \alpha_2 p(\Sigma(\beta,\alpha),\phi v) \\ &\quad + \alpha_2 p(\Sigma(\upsilon,\nu),\phi\alpha) + \alpha_4 p(\Sigma(\upsilon,u),\phi\beta), \end{aligned} \tag{39}$$

for every pairs $(\alpha, \beta), (\upsilon, \nu) \in \Omega \times \Omega$ *such that* $\phi\alpha \preceq \phi u$ *and* $\phi\beta \succeq \phi v$*, where*
(1) $\alpha_3 + \alpha_4 + \sum_{i=1}^{4} \alpha_i < 1$*, if* $\alpha_1 - \alpha_3 < 0$ *and* $\alpha_2 - \alpha_4 < 0$*;*
(2) $\alpha_4 + \sum_{i=1}^{4} \alpha_i < 1$*, if* $\alpha_1 - \alpha_3 \geq 0$ *and* $\alpha_2 - \alpha_4 < 0$*;*
(3) $\alpha_3 + \sum_{i=1}^{4} \alpha_i < 1$*, if* $\alpha_1 - \alpha_3 < 0$ *and* $\alpha_2 - \alpha_4 \geq 0$*;*
(4) $\sum_{i=1}^{4} \alpha_i < 1$*, if* $\alpha_1 - \alpha_3 \geq 0$ *and* $\alpha_2 - \alpha_4 \geq 0$*.*

Suppose that
(1) $\Sigma(\Omega \times \Omega) \subseteq \phi(\Omega)$*;*
(2) Σ *has the mixed* ϕ*-monotone property;*
(3) ϕ *is continuous and monotone increasing and* Σ *and* g *be partial-compatible mappings.*

Also, suppose that
(a) Σ *is continuous, or,*
(b) Ω *has the following properties:*
(i) *if* $\{\alpha_n\}$ *is a nondecreasing sequence and* $\alpha \in \Omega$ *with* $\lim_{n\to\infty} p(\alpha_n, \alpha) = p(\alpha, \alpha) = 0$*, then* $\alpha_n \preceq \alpha$ *for all* n*,*
(ii) *if* $\{\alpha_n\}$ *is a nonincreasing sequence and* $\alpha \in \Omega$ *with* $\lim_{n\to\infty} p(\alpha_n, \alpha) = p(\alpha, \alpha) = 0$*, then* $\alpha \preceq \alpha_n$ *for all* n*.*

If there exist $\alpha_0, \beta_0 \in \Omega$ *such that* $\phi\alpha_0 \preceq \Sigma(\alpha_0, \beta_0)$ *and* $\Sigma(\beta_0, \alpha_0) \preceq \phi beta_0$*, then* Σ *and* ϕ *have a coupled coincidence point in* Ω*.*

According to [7, Example 3.1], we present the following example to illustrate our results.

Example 2.2. Let $\Omega = [0, 1]$. Then $(\Omega, \leq)$ is a partially ordered set with the natural ordering of real numbers. Let $p(\alpha, \beta) = \max\{\alpha, \beta\}$ for all $\alpha, \beta \in [0, 1]$. Then

(Ω, p) is a complete partial metric space. Let $\phi : \Omega \to \Omega$ be defined as $\phi(\alpha) = \alpha^2$, for all $\alpha \in \Omega$ and $\Sigma : \Omega \times \Omega \longrightarrow \Omega$ be defined as

$$\Sigma(\alpha, \beta) = \begin{cases} \frac{\alpha^2-\beta^2}{12}, & \text{if } \alpha \geq \beta, \\ 0, & \text{if } \alpha < \beta. \end{cases}$$

Obviously, Σ enjoys the mixed g-monotone property. As in Example 3.1 of [7], we can show that the mappings Σ and ϕ are partial-compatible in Ω.

We next verify inequality (1) of Theorem 2.1. We take $\alpha, \beta, u, v \in \Omega$ such that $\phi\alpha \leq gu$ and $\phi\beta \geq \phi v$, that is, $\alpha^2 \leq u^2$ and $\beta^2 \geq v^2$. We consider the following cases:

Case 1: $\alpha \geq \beta$ and $u \geq v$:

$$\begin{aligned} &p(\Sigma(\alpha,\beta),\Sigma(\upsilon,\nu)) = \max\{\tfrac{\alpha^2-\beta^2}{12}, \tfrac{u^2-v^2}{12}\} = \tfrac{u^2-v^2}{12} \\ &\leq \tfrac{1}{11}u^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}\alpha^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}u^2 + \tfrac{1}{11}v^2 \\ &\quad + \tfrac{1}{11}\max\{\tfrac{u^2-v^2}{3}, \alpha^2\} + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}u^2 + \tfrac{1}{11}v^2 \\ &= \alpha_1 p(\phi\alpha, \phi u) + \alpha_2 p(\phi\beta, \phi v) + \alpha_3 p(\Sigma(\alpha,\beta), \phi\alpha) + \alpha_4 p(\Sigma(\beta,\alpha), gy) \\ &\quad + \alpha_5 p(\Sigma(\alpha,\beta), \phi u) + \alpha_6 p(\Sigma(\beta,\alpha), \phi v) + \alpha_7 p(\Sigma(\upsilon,\nu), \phi\alpha) \\ &\quad + \alpha_8 p(\Sigma(v,u), \phi\beta) + \alpha_9 p(\Sigma(\upsilon,\nu), \phi u) + \alpha_{10} p(\Sigma(v,u), \phi v), \end{aligned} \tag{40}$$

Case 2: $\alpha \geq \beta$ and $u < v$:

$$\begin{aligned} &p(\Sigma(\alpha,\beta),\Sigma(\upsilon,\nu)) = \max\{\tfrac{\alpha^2-\beta^2}{12}, 0\} = \tfrac{\alpha^2-\beta^2}{12} \\ &\leq \tfrac{1}{11}u^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}\alpha^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}u^2 + \tfrac{1}{11}v^2 \\ &\quad + \tfrac{1}{11}\alpha^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}u^2 + \tfrac{1}{11}v^2 \\ &= \alpha_1 p(\phi\alpha, \phi u) + \alpha_2 p(\phi\beta, \phi v) + \alpha_3 p(\Sigma(\alpha,\beta), \phi\alpha) + \alpha_4 p(\Sigma(\beta,\alpha), \phi\beta) \\ &\quad + \alpha_5 p(\Sigma(\alpha,\beta), \phi u) + \alpha_6 p(\Sigma(\beta,\alpha), \phi v) + \alpha_7 p(\Sigma(\upsilon,\nu), \phi\alpha) \\ &\quad + \alpha_8 p(\Sigma(v,u), \phi\beta) + \alpha_9 p(\Sigma(\upsilon,\nu), \phi u) + \alpha_{10} p(\Sigma(v,u), \phi v), \end{aligned} \tag{41}$$

Case 3: $\alpha < \beta$ and $u \geq v$:

$$\begin{aligned} &p(\Sigma(\alpha,\beta),\Sigma(\upsilon,\nu)) = \max\{0, \tfrac{u^2-v^2}{12}\} = \tfrac{u^2-v^2}{12} \\ &\leq \tfrac{1}{11}u^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}\alpha^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}u^2 + \tfrac{1}{11}\max\{\tfrac{\beta^2-\alpha^2}{3}, v^2\} \\ &\quad + \max\{\tfrac{u^2-v^2}{3}, \alpha^2\} + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}u^2 + \tfrac{1}{11}v^2 \\ &= \alpha_1 p(\phi\alpha, \phi u) + \alpha_2 p(\phi\beta, \phi v) + \alpha_3 p(\Sigma(\alpha,\beta), \phi\alpha) + \alpha_4 p(\Sigma(\beta,\alpha), \phi\beta) \\ &\quad + \alpha_5 p(\Sigma(\alpha,\beta), \phi u) + \alpha_6 p(\Sigma(\beta,\alpha), \phi v) + \alpha_7 p(\Sigma(\upsilon,\nu), \phi\alpha) \\ &\quad + \alpha_8 p(\Sigma(v,u), \phi\beta) + \alpha_9 p(\Sigma(\upsilon,\nu), \phi u) + \alpha_{10} p(\Sigma(v,u), \phi v), \end{aligned} \tag{42}$$

Case 4: $\alpha < \beta$ and $u < v$:

$$\begin{aligned} &p(\Sigma(\alpha,\beta),\Sigma(\upsilon,\nu)) = \max\{0, 0\} = 0 \\ &\tfrac{1}{11}u^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}\alpha^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}u^2 + \tfrac{1}{11}\max\{\tfrac{\beta^2-\alpha^2}{3}, v^2\} \\ &+ \tfrac{1}{11}\alpha^2 + \tfrac{1}{11}\beta^2 + \tfrac{1}{11}u^2 + \tfrac{1}{11}v^2 \\ &= \alpha_1 p(\phi\alpha, \phi u) + \alpha_2 p(\phi\beta, \phi v) + \alpha_3 p(\Sigma(\alpha,\beta), \phi\alpha) + \alpha_4 p(\Sigma(\beta,\alpha), \phi\beta) \\ &\quad + \alpha_5 p(\Sigma(\alpha,\beta), \phi u) + \alpha_6 p(\Sigma(\beta,\alpha), \phi v) + \alpha_7 p(\Sigma(\upsilon,\nu), \phi\alpha) \\ &\quad + \alpha_8 p(\Sigma(v,u), \phi\beta) + \alpha_9 p(\Sigma(\upsilon,\nu), \phi u) + \alpha_{10} p(\Sigma(v,u), \phi v). \end{aligned} \tag{43}$$

Thus, it is verified that Σ and ϕ satisfy all the conditions of Theorem 2.1 with $\alpha_i = \frac{1}{11}$, for all $1 \leq i \leq 10$. Here $(0,0)$ is the coupled coincidence point of Σ and ϕ in Ω.

Acknowledgment

Authors are thankful to the editor and anonymous referees for their valuable comments and suggestions.

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Received: May 2021
Accepted: June 2021