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Analytic differenceability of functions

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Abstract. Analytic summability of functions was introduced by Hooshmand in 2016. He used Bernoulli numbers and polynomials $B_n(z)$ to define analytic summability and related analytic summand functions. Since the Bernoulli and Euler polynomials have many similarities, so it motivated us to define differenceability and introduce analytic difference function of a complex or real function by utilizing the Euler numbers and polynomials $E_n(z)$. Also, we prove some criteria for analytic differenceability of analytic functions. Moreover, we observe that the analytic difference function is indeed a series of the Euler polynomials and arrive at some series convergence tests for Euler polynomial series $\sum_{n=0}^{\infty} c_n E_n(z)$.

1. Introduction and preliminaries

Webster in [11], studied gamma type functions satisfying the functional equation $f(x+1) = g(x)f(x)$ (for a given $g : \mathbb{R}^+ \to \mathbb{R}^+$) and obtained a generalization of the Bohr-Mollerup theorem ([4]) in 1997. On the other hand, Hooshmand in 2001 introduced a new concept entitled limit summability of functions and their summand functions for every function defined on a subset of $\mathbb{R}$ or $\mathbb{C}$ containing positive integers. Also, He showed that the topic of gamma type functions is a subtopic of limit summability and concluded some generalizations of many their properties.

In addition, in 2010, Muller and Schleicher introduced the concept of fractional sums and Euler-like identities in [9]. In fact, they arrived at the functional sequence $f_{\sigma_n}(x)$ introduced by Hooshmand (of course in the special case $\sigma = 0$) while they were not aware of the limit summability topic, even though they did not notice at theorems or conditions of convergence of the functional sequence.

It is worth noting that some of the well known functions such as exponential, hyperbolic and trigonometric functions are not limited summable. In 2016, Hooshmand introduced another type of summability entitled analytic summability of functions [7] to remove this problem. To this purpose, he applied Bernoulli polynomials and

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numbers to define analytic summability. Moreover, in 2017, he introduced functional sequential and trigonometric summability of functions which is a generalization of analytic summability. Recently, he and his colleagues improved some upper bounds and inequalities for analytic summand functions (see [8]).

In this regard, we define *analytic differenceability* of functions by motivating from analytic summability. Also, we introduce the *analytic difference function* of a complex or real function. Moreover, we prove some criteria for analytic differenceability. In this way we obtain some series convergence tests for Euler polynomials series $\sum_{n=0}^{\infty} c_n E_n(z)$.

First, we recall some definitions and properties of Bernoulli and Euler polynomials $B_n(z)$ and $E_n(z)$ $(n = 0, 1, 2, ...; z \in \mathbb{C})$ of [1, 3] as follows

$$\frac{te^{zt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!} \quad (|t| < 2\pi) \quad \text{and} \quad \frac{2e^{zt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!} \quad (|t| < 2\pi). \tag{1}$$

The first few polynomials are

$$B_0(z) = 1, \quad B_1(z) = z - \frac{1}{2}, \quad B_2(z) = z^2 - z + \frac{1}{6}, \quad B_3(z) = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z,$$
$$E_0(z) = 1, \quad E_1(z) = z - \frac{1}{2}, \quad E_2(z) = z^2 - z, \quad E_3(z) = z^3 - \frac{3}{2}z^2 + \frac{1}{4}.$$

The numbers $B_n = B_n(0)$ and $E_n = 2^n E_n(\frac{1}{2})$ are Bernoulli and Euler numbers. All Bernoulli numbers are rational and all Euler numbers are integers. Explicit formula for $B_n(z)$ and $E_n(z)$ are given by

$$B_n(z) = \sum_{k=0}^{n} \binom{n}{k} B_k z^{n-k} \quad \text{and} \quad E_n(z) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} (z - \frac{1}{2})^{n-k} \tag{2}$$

Also, the Bernoulli and Euler polynomials follow many relations, for example

$$B_n(z+1) - B_n(z) = nz^{n-1} \quad \text{and} \quad E_n(z+1) + E_n(z) = 2z^n \tag{3}$$

The second Bernoulli numbers are denoted by $b_n = B_n(1)$.
Now, we recall notations, basic definition and some main theorems of analytic summability introduced by Hooshmand in [7].

$$\sigma(z^n) = S_n(z) = \frac{B_{n+1}(z+1) - b_{n+1}}{n+1}, \quad z \in \mathbb{C}, n \geq 0, \tag{4}$$

$$\beta_{nk} = \beta_{n,k} := \binom{n+1}{k} \frac{b_{n+1-k}}{n+1} = \frac{n!}{k!(n+1-k)!} b_{n+1-k}, \tag{5}$$

$$S_n(z) = \sum_{k=1}^{n+1} \beta_{nk} z^k, \quad z \in \mathbb{C}, n \geq 0. \tag{6}$$

Note. It is worth noting that $\sigma(z^n)$ is just a formal symbol and we can not put a special z in it directly, so for this purpose we should apply $S_n(z)$ (see [7]).

Definition 1.1. *Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$ be a complex or real analytic function defined on an open domain D. We call f* analytic summable *at z_0 (resp. absolutely analytic summable) if the series*

$$f_{\sigma_A}(z_0)=f_\sigma(z_0)=\sum_{n=0}^{\infty}c_n\sigma(z^n)=\sum_{n=0}^{\infty}c_nS_n(z),$$

is convergent (resp. absolutely convergent). We call f analytic summable on $E\subseteq D$ if it is analytic summable at every point of E. The function $f_{\sigma_A}=f_\sigma$ (with the largest possible domain) is called analytic summand (function) of f. If f is analytic summable on the whole $\mathbb{C}$, then we call f entire summable.

If f is analytic summable on D then f_σ satisfies the well-known difference functional equation (e.g., see [5]) below

$$f_\sigma(z)=f(z)+f_\sigma(z-1)\quad;\quad z\in D\cap D+1. \tag{7}$$

Also, some upper bounds and criteria for analytic summand functions are obtained by applying the following bounds from [2, 6, 10].

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{-2n}}<|B_{2n}|<\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{1-2n}}, \tag{8}$$

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{\alpha-2n}}<|B_{2n}|<\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{\beta-2n}}, \tag{9}$$

where $\beta=2+\frac{\ln(1-\frac{6}{\pi^2})}{\ln(2)}\approx 0.6491...$.

$$\frac{2(2n)!}{\pi^{2n}(2^{2n}-1)}<|B_{2n}|<\frac{2(2^{2j}-1)}{2^{2j}}\zeta(2j)\frac{2(2n)!}{\pi^{2n}(2^{2n}-1)}, \tag{10}$$

where ζ is the zeta Rimman function, j is a fixed positive integer and $n\geq j$.

Here we summarize some of them as a theorem. Assume that $f(z)=\sum_{n=0}^{\infty}c_nz^n$ and put

$$Abs(f(z))=\sum_{n=0}^{\infty}|c_n||z|^n,\quad Abs^{e}_{\frac{1}{\pi}}(f)=\sum_{\substack{n=0\\ n\text{ is even}}}^{\infty}\frac{n!}{\pi^n}|c_n|,\quad Abs^{o}_{\frac{1}{\pi}}(f)=\sum_{\substack{n=0\\ n\text{ is odd}}}^{\infty}\frac{n!}{\pi^n}|c_n|.$$

Theorem 1.2. *Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$ be an analytic function, if the series $\sum_{n=0}^{\infty}\frac{n!}{\pi^n}c_n$ is absolutely convergent, then f is absolutely entirely summable. Moreover, the analytic summand function f_σ is analytic on $\mathbb{C}$ and represent as follows*

$$f_\sigma(z)=\sum_{n=1}^{\infty}\sigma_nz^n=\sum_{n=1}^{\infty}\frac{1}{n!}\left(\sum_{j=0}^{\infty}\frac{(j+n-1)!}{j!}b_jc_{j+n-1}\right)z^n,\quad z\in\mathbb{C}, \tag{11}$$

where $\sigma_n := \lim_{N\to\infty} \sigma_{n,N} = \sum_{k=n-1}^{\infty} \beta_{k,n} c_k$.

In addition, f_σ *satisfies the following inequalities*

$$\begin{aligned}
|f_\sigma(z)| &\le \frac{1}{2}Abs(f(z)) + \frac{1}{3}Abs(F(z)) + \frac{2}{3\pi}\sinh(\pi|z|)Abs^e_{\frac{1}{\pi}}(f)\\
&+ \frac{2}{3\pi}(\cosh(\pi|z|)-1)Abs^o_{\frac{1}{\pi}}(f) \qquad (12)\\
&\le \frac{1}{2}Abs(f(z)) + \frac{12-\pi^2}{12}Abs(F(z)) + \frac{\pi}{12}\sinh(\pi|z|)Abs^e_{\frac{1}{\pi}}(f)\\
&+ \frac{\mu}{\pi}(\cosh(\pi|z|)-1)Abs^o_{\frac{1}{\pi}}(f)\\
&\le \frac{2}{\pi}(e^{\pi|z|}-1)Abs_{\frac{1}{\pi}}(f); \quad z\in\mathbb{C}.
\end{aligned}$$

where $F(z) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1}$.

Proof. See Theorem 4.1 of [7] and Corollary 2.2 of [8]. □

2. Induced analytic differenceability from analytic summability of functions

Now we can define the analytic differenceability for analytic functions motivated by analytic summability topic. At this point, first, we introduce the notation of analytic differenceability of complex and real functions as follows. Followed by the notations in the analytic summability topic in [7], we put

$$\delta_A(z^n) = \delta(z^n) := E_n(z).$$

Since $E_n(z+1) + E_n(z) = 2z^n$ $(z\in\mathbb{C}, n\ge 0)$ then

$$E_n(m) = 2\sum_{k=1}^{m-1}(-1)^{m-k-1}k^n + (-1)^m E_n(0), \quad m\in\mathbb{N},$$

and

$$\delta(z^n) = 2z^n - \delta((z-1)^n) \quad ; \quad z\in\mathbb{C}, n\ge 0. \qquad (13)$$

Definition 2.1. *Let* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *be a complex or real analytic function defined on open domain* D*. We say that* f *is analytic differenceable at* z_0 *(resp. absolutely analytic differenceable) if the series*

$$f_\delta(z) = \sum_{n=0}^{\infty} c_n \delta(z^n) = \sum_{n=0}^{\infty} c_n E_n(z)$$

is convergent (resp. absolutely convergent). We say that f *is analytic differenceable on* $E\subseteq D$ *if it is analytic differenceable at every point of* E*. If* f *is analytic differenceable on the whole* $\mathbb{C}$*, then we say that* f *is entire differenceable.*

Similar to the mentioned for analytic summability, $\delta(z^n)$ is just a formal symbol and we can not put a special z in it directly.

Note . We apply the following identity for iterated series of double complex sequences, which represents the sum of all arrays of the lower triangle of the $N \times N$ matrix $[C_{nk}]$ by two different ways:

$$\sum_{n=0}^{N}\sum_{k=0}^{n} C_{nk} = \sum_{n=0}^{N}\sum_{k=n}^{N} C_{kn} \tag{14}$$

Example 2.2. *The exponential function* $\exp(z) = e^z$ *is entire differenceable and* $\exp_\delta(z) = \frac{2e^z}{e+1}$, $z \in \mathbb{C}$. *By applying (14) we can write*

$$\begin{aligned}\exp_\delta(z) = \sum_{n=0}^{\infty}\frac{1}{n!}E_n(z) &= \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{k=0}^{n}\frac{1}{n!}\frac{n!}{k!(n-k)!}\frac{E_{n-k}}{2^{n-k}}(z-\frac{1}{2})^k \\ &= \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{k=n}^{N}\frac{1}{n!}\frac{E_{k-n}}{2^{k-n}}(z-\frac{1}{2})^n = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\frac{E_j}{j!}(\frac{1}{2})^j(z-\frac{1}{2})^n \\ &= \frac{2\sqrt{e}}{e+1}\sum_{n=0}^{\infty}\frac{(z-\frac{1}{2})^n}{n!} = \frac{2\sqrt{e}}{e+1}e^{(z-\frac{1}{2})} = \frac{2e^z}{e+1}.\end{aligned}$$

Now we state some basic properties of analytic differenceability of functions as follows.

Theorem 2.3. *Let* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} d_n z^n$ *be analytic functions on an open domain* D.
(a) If $z, z-1 \in D$, *then* f *is analytic differenceable at* z *if and only if it is analytic differenceable at* $z-1$. *So if* f *is analytic differenceable on* D *then*

$$f_\delta(z) = 2f(z) - f_\delta(z-1) \quad ; \quad z \in D \cap D+1. \tag{15}$$

(b) If f *is analytic differenceable on* D *and* $D \subseteq D+1$, *then*

$$f_\delta(z) = 2f(z) - f_\delta(z-1) \quad ; \quad z \in D. \tag{16}$$

(c) If f, g *are analytic differenceable at* z *(resp. on* D*), then every linear combination of* f, g *is so and we have* $(af+bg)_\delta(z) = af_\delta(z) + bg_\delta(z)$ *for all* $z \in D$.

Proof. Let $f_{\delta_N}(z) := \sum_{n=0}^{N} c_n \delta(z^n)$. If $z, z-1 \in D$, then by applying (13) we have

$$f_{\delta_N}(z) = 2\sum_{n=0}^{N} c_n z^n + f_{\delta_N}(z-1).$$

Moreover, a simple calculation indicates that

$$(af+bg)_{\delta_N}(z) = af_{\delta_N}(z) + bg_{\delta_N}(z).$$

Now getting $N \to \infty$ implies the result. □

3. Upper bounds for Euler polynomials

The analytic difference function is defined by Euler polynomials $E_n(z)$, so finding upper bounds for $E_n(z)$ plays an important role to prove some criteria for analytic differenceability. First, we recall the following upper bounds for Euler numbers from [1].

$$\frac{4^{r+1}(2r)!}{\pi^{2r+1}(1+3^{-1-2r})} < |E_{2r}| < \frac{4^{r+1}(2r)!}{\pi^{2r+1}} \tag{17}$$

Since $E_{2r+1} = 0$ together (17) imply that

$$|E_n| < \frac{4^{(\frac{n}{2}+1)}n!}{\pi^{n+1}}.$$

Now we put

$$\eta_{n,k} := \binom{n}{k}\frac{E_{n-k}}{2^{n-k}} \quad ; \quad n \geq 0,\ 0 \leq k \leq n. \tag{18}$$

Therefore we have

$$|\eta_{n,k}| < \frac{n!}{k!(n-k)!}\frac{4^{(\frac{n-k}{2}+1)}(n-k)!}{\pi^{n-k+1}} = \frac{4n!}{k!\pi^{n-k+1}} \quad ; \quad 0 \leq k \leq n. \tag{19}$$

Hence

$$|\eta_{n,k}| \leq \begin{cases} 1 & ; n = k \\ 0 & ; n-k \quad \text{is odd} \\ \frac{4n!}{k!\pi^{n-k+1}} & ; n-k \quad \text{is even.} \end{cases}$$

Applying (2) and (18) we have

$$\begin{aligned} |E_n(z)| = |\sum_{k=0}^{n} \eta_{n,k}(z-\frac{1}{2})^k| &\leq \sum_{k=0}^{n} \frac{4n!}{k!\pi^{n-k+1}}|(z-\frac{1}{2})|^k \\ &= \frac{4n!}{\pi^{n+1}}\sum_{k=0}^{n}\frac{\pi^k}{k!}|(z-\frac{1}{2})|^k \\ &\leq \frac{4n!}{\pi^{n+1}}e^{\pi|z-\frac{1}{2}|} \end{aligned} \tag{20}$$

Also we can write

$$\begin{aligned}|E_n(z)| &= |\sum_{k=0}^{n}\eta_{n,k}(z-\frac{1}{2})^k| \\ &\leq \sum_{k=0}^{n}|\eta_{n,k}||z-\frac{1}{2}|^k = |z-\frac{1}{2}|^n + \sum_{\substack{k=0\\ n-k \text{ is even}}}^{n-1}\eta_{n,k}(z-\frac{1}{2})^k| \\ &= |z-\frac{1}{2}|^n + \begin{cases}\sum_{m=0}^{\frac{n}{2}}|\eta_{n,2m}||z-\frac{1}{2}|^{2m} & ;n \quad \text{is even} \\ \sum_{m=1}^{\frac{n-1}{2}}|\eta_{n,2m-1}||z-\frac{1}{2}|^{2m-1} & ;n \quad \text{is odd.}\end{cases} \qquad (21)\end{aligned}$$

4. Some criteria for analytic differenceability of functions

In this section, we prove some criteria for analytic differnecability by applying the obtained upper bounds for $E_n(z)$ along with previous results. Before starting the proofs we should do attention to a convention below.

Convention. We put $\eta_{n,-1} = 0$ for every $n \in \mathbb{N}$ and if $-1!$ appears in a denominator of a fraction then that fraction will equal to zero.

Theorem 4.1. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an analytic function. If $\sum_{n=0}^{\infty}\frac{n!}{\pi^n}c_n$ is absolutely convergent then f is absolutely entirely differenceable. Moreover, the analytic difference function f_δ is an entire function by representation below*

$$f_\delta(z) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\frac{(j+n)!}{j!}\frac{E_j}{2^j}c_{j+n}(z-\frac{1}{2})^n, \ z \in \mathbb{C}. \qquad (22)$$

In addition, the following upper bounds for f_δ hold:

$$\begin{aligned}|f_\delta(z)| &\leq Abs(f(z-\frac{1}{2})) + \frac{4}{\pi}\cosh(\pi|z-\frac{1}{2}|)Abs^e_{\frac{1}{\pi}}(f) \\ &+ \frac{4}{\pi}\sinh(\pi|z-\frac{1}{2}|)Abs^o_{\frac{1}{\pi}}(f) \\ &\leq \frac{4e^{\pi|z-\frac{1}{2}|}}{\pi}Abs_{\frac{i}{\pi}}(f), \ z \in \mathbb{C}. \qquad (23)\end{aligned}$$

PROOF. Firstly we show that the function f is an entire function. Without loss of generality of proof, let $c_n \neq 0$ for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\frac{(\pi|z|)^n}{n!} = 0$, for every $z \in \mathbb{C}$, the limit comparison test between $\sum_{n=0}^{\infty}\frac{n!}{\pi^n}|c_n|$ and $\sum_{n=0}^{\infty}|c_n|z^n$ implies that the series $\sum_{n=0}^{\infty}|c_n|z^n$ is convergent on $\mathbb{C}$.

Now we use (21) to prove the first inequality as follows

$$
\begin{aligned}
|f_{\delta_N}(z)| = |\sum_{n=0}^{N} c_n E_n(z)| &\leq \sum_{n=0}^{N} |c_n||E_n(z)| \\
&= \sum_{\substack{n=0 \\ n \text{ is even}}}^{N} |c_n|\big(|z-\frac{1}{2}|^n + \sum_{m=0}^{\frac{n}{2}} |\eta_{n,2m}||z-\frac{1}{2}|^{2m}\big) \\
&+ \sum_{\substack{n=0 \\ n \text{ is odd}}}^{N} |c_n|\big(|z-\frac{1}{2}|^n + \sum_{m=0}^{\frac{n-1}{2}} |\eta_{n,2m-1}||z-\frac{1}{2}|^{2m-1}\big) \\
&\leq \sum_{\substack{n=0 \\ n \text{ is even}}}^{N} |c_n|\big(|z-\frac{1}{2}|^n + \sum_{m=0}^{\frac{n}{2}} \frac{4n!}{(2m)!\pi^{n-2m+1}}|z-\frac{1}{2}|^{2m}\big) \\
&+ \sum_{\substack{n=0 \\ n \text{ is odd}}}^{N} |c_n|\big(|z-\frac{1}{2}|^n + \sum_{m=0}^{\frac{n-1}{2}} \frac{4n!}{(2m-1)!\pi^{n-2m+2}}|z-\frac{1}{2}|^{2m-1}\big) \\
&= \sum_{n=0}^{N} |c_n||z-\frac{1}{2}|^n + \sum_{\substack{n=0 \\ n \text{ is even}}}^{N} |c_n|\frac{4n!}{\pi^{n+1}} \sum_{m=0}^{\frac{n}{2}} \frac{\pi^{2m}|z-\frac{1}{2}|^{2m}}{(2m)!} \\
&+ \sum_{\substack{n=0 \\ n \text{ is odd}}}^{N} |c_n|\frac{4n!}{\pi^{n+1}} \sum_{m=0}^{\frac{n-1}{2}} \frac{\pi^{2m-1}|z-\frac{1}{2}|^{2m-1}}{(2m-1)!} \\
&\leq \sum_{n=0}^{\infty} |c_n||z-\frac{1}{2}|^n + \frac{4}{\pi}\cosh(\pi|z-\frac{1}{2}|) \sum_{n=0}^{\infty} \frac{n!}{\pi^n}|c_n| \\
&+ \frac{4}{\pi}\sinh(\pi|z-\frac{1}{2}|) \sum_{n=0}^{\infty} \frac{n!}{\pi^n}|c_n| \\
&= Abs_N(f(z-\frac{1}{2})) + \frac{4}{\pi}\cosh(\pi|z-\frac{1}{2}|)Abs^{e}_{\frac{!}{\pi}N}(f) \\
&+ \frac{4}{\pi}\sinh(\pi|z-\frac{1}{2}|)Abs^{o}_{\frac{!}{\pi}N}(f).
\end{aligned}
$$

Now letting $N \to \infty$ implies the result. Also by applying (20) we have

$$
\begin{aligned}
|f_{\delta_N}(z)| &\leq \sum_{n=0}^{N} |c_n|\frac{4n!}{\pi^{n+1}}e^{\pi|z-\frac{1}{2}|} \leq \frac{4e^{\pi|z-\frac{1}{2}|}}{\pi} \sum_{n=0}^{\infty} \frac{|c_n|}{\pi^n}n! \\
&= \frac{4e^{\pi|z-\frac{1}{2}|}}{\pi} Abs_{\frac{i}{\pi}N}(f).
\end{aligned}
$$

Letting $N\to\infty$ implies the second inequality. Now, by applying (2) and (18) we have

$$\begin{aligned}f_\delta(z) &= \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{k=0}^{n}c_n\eta_{n,k}(z-\frac{1}{2})^k = \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{k=n}^{N}c_k\eta_{k,n}(z-\frac{1}{2})^n\\ &= \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{k=n}^{N}c_k\frac{k!}{n!(k-n)!}\frac{E_{k-n}}{2^{k-n}}(z-\frac{1}{2})^n\\ &= \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{j=0}^{N}c_{j+n}\frac{(j+n)!}{n!j!}\frac{E_j}{2^j}(z-\frac{1}{2})^n\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\frac{(j+n)!}{j!}\frac{E_j}{2^j}c_{j+n}(z-\frac{1}{2})^n.\end{aligned}$$

□

Corollary 4.2. *Let c_n be a complex or real sequence such that $\sum_{n=0}^{\infty}\frac{n!}{\pi^n}c_n$ is absolutely convergent then the series $\sum_{n=0}^{\infty}c_nE_n(z)$ is absolutely convergent on $\mathbb{C}$. Moreover, the series has an analytic answer which is represented as follows*

$$\sum_{n=0}^{\infty}c_nE_n(z)=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\frac{(j+n)!}{j!}\frac{E_j}{2^j}c_{j+n}(z-\frac{1}{2})^n. \tag{24}$$

In addition, the series $\sum_{n=0}^{\infty}c_nE_n(z)$ satisfies the following inequalities:

$$\begin{aligned}|\sum_{n=0}^{\infty}c_nE_n(z)| &\le Abs(f(z-\frac{1}{2}))+\frac{4}{\pi}\cosh(\pi|z-\frac{1}{2}|)Abs^{e}_{\frac{1}{\pi}}(f)\\ &\quad+\frac{4}{\pi}\sinh(\pi|z-\frac{1}{2}|)Abs^{o}_{\frac{1}{\pi}}(f)\\ &\le\frac{4e^{\pi|z-\frac{1}{2}|}}{\pi}Abs_{\frac{i}{\pi}}(f)\end{aligned}$$

Proof. See Theorem 4.1. □

Corollary 4.3. *If $\sum_{n=0}^{\infty}\frac{n!}{\pi^n}c_n$ is absolutely convergent then the function $f(z)=\sum_{n=0}^{\infty}c_nz^n$ is absolutely entirely summable and absolutely entirely differenceable simultaneously.*

Corollary 4.4. *Analytic difference function of every polynomial of degree n exists and it is a polynomial of the same degree.*

Theorem 4.5. *Let $\sum_{n=0}^{\infty}c_nz^n$ be a complex or real analytic function. If $\sqrt[n]{n!|c_n|}\le\theta<\pi$ for all n then f is absolutely entirely differenceable, and the following inequality holds:*

$$|f_\delta(z)|\le\frac{4}{\pi-\theta}e^{\pi|z-\frac{1}{2}|}\quad;\quad z\in D \tag{25}$$

Proof.

$$|f_\delta(z)| \leq \sum_{n=0}^{\infty} |c_n||z^n| \leq \sum_{n=0}^{\infty} |c_n| \frac{4}{\pi} e^{\pi|z-\frac{1}{2}|} \frac{n!}{\pi^n} \leq \frac{4e^{\pi|z-\frac{1}{2}|}}{\pi} \sum_{n=0}^{\infty} (\frac{\theta}{\pi})^n$$
$$= \frac{4}{\pi-\theta} e^{\pi|z-\frac{1}{2}|}.$$

□

5. Applications

As it mentioned, most of analytic summable functions such as polynomials, trigonometric and exponential functions are analytic differenceable too. Here we find their analytic difference functions. Moreover, we obtain upper bounds for differences of powers of natural numbers.

Proposition 5.1. *Let $r \in \mathbb{N}$, $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and the series $\sum_{n=0}^{\infty} \frac{n!}{\pi^n} c_n$ be absolutely convergent, then the partial summation $\sum_{k=1}^{r} (-1)^{r-k} f(k)$ satisfies the following inequalities*

$$|\sum_{k=1}^{r-1} (-1)^{r-k-1} f(k)| \leq \frac{1}{2}\Big(Abs(f(r-\frac{1}{2})) + \frac{4}{\pi}\cosh(\pi|r-\frac{1}{2}|) Abs^e_{\frac{1}{\pi}}(f)$$
$$+ \frac{4}{\pi}\sinh(\pi|r-\frac{1}{2}|) Abs^o_{\frac{1}{\pi}}(f) + (-1)^{r-1} f_\delta(0)\Big)$$
$$\leq \frac{1}{2}\Big(\frac{4e^{\pi|r-\frac{1}{2}|}}{\pi} Abs_{\frac{i}{\pi}}(f) + (-1)^{r-1} f_\delta(0)\Big), \ z \in \mathbb{C}. \qquad (26)$$

Proof. Since $f(z)$ is absolutely entirely differenceable, then functional equation (15) implies that

$$f_\delta(r) = 2\sum_{k=1}^{r-1} (-1)^{r-k-1} f(k) + (-1)^r f_\delta(0), \quad r \in \mathbb{N}.$$

Now, applying Theorem 4.1 for $z = r$ gets the result. □

Example 5.2. *If $p \in \mathbb{N}$, the function $f(z) = z^p$, is absolutely entirely differenceable and the following upper bounds for r difference-sum of power of natural numbers hold*
If p is even

$$(-1)^{r-1}1^p + (-1)^{r-2}2^p + \ldots - (r-1)^p + r^p \leq \frac{1}{2}\Big(|r-\frac{1}{2}| + \frac{4p!}{\pi^{p+1}}\cosh(|r-\frac{1}{2}|)$$
$$+ (-1)^{r-1} E_p(0)\Big)$$
$$\leq \frac{1}{2}\Big(\frac{4p!}{\pi^{p+1}} e^{\pi|r-\frac{1}{2}|} + (-1)^{r-1} E_p(0)\Big), \ z \in \mathbb{C}.$$

Particularly, if p is even and r is odd we have

$$\begin{aligned}1^p - 2^p + 3^p ... - (r-1)^p + r^p &\leq \frac{1}{2}\Big(|r - \frac{1}{2}| + \frac{4p!}{\pi^{p+1}}\cosh(|r-\frac{1}{2}|)\\ &\quad + E_p(0)\Big)\\ &\leq \frac{1}{2}\Big(\frac{4p!}{\pi^{p+1}}e^{\pi|r-\frac{1}{2}|} + E_p(0)\Big),\ z\in\mathbb{C}.\end{aligned}$$

If p is odd

$$\begin{aligned}(-1)^{r-1}1^p + (-1)^{r-2}2^p + ... - (r-1)^p + r^p &\leq \frac{1}{2}\Big(|r - \frac{1}{2}| + \frac{4p!}{\pi^{p+1}}\sinh(|r-\frac{1}{2}|)\Big)\\ &\leq \frac{1}{2}\Big(\frac{4p!}{\pi^{p+1}}e^{\pi|r-\frac{1}{2}|}\Big),\ z\in\mathbb{C}.\end{aligned}$$

Particularly, if p and r are odd we have

$$\begin{aligned}1^p - 2^p + 3^p ... - (r-1)^p + r^p &\leq \frac{1}{2}\Big(|r - \frac{1}{2}| + \frac{4p!}{\pi^{p+1}}\sinh(|r-\frac{1}{2}|)\Big)\\ &\leq \frac{1}{2}\Big(\frac{4p!}{\pi^{p+1}}e^{\pi|r-\frac{1}{2}|}\Big),\ z\in\mathbb{C}.\end{aligned}$$

Example 5.3. *The function $f(z) = a^z$ is entire differenceable if $|\ln a| < \pi$ (see Theorem 4.1), and by (22) we have*

$$\begin{aligned}\delta(a^z) = \sum_{n=0}^{\infty}\frac{(\ln a)^n}{n!}E_n(z) &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\frac{(j+n)!}{j!}\frac{E_j}{2^j}\frac{(\ln a)^{n+j}}{(j+n)!}\\ &= \sum_{n=0}^{\infty}\frac{(\ln a)^n}{n!}\sum_{j=0}^{\infty}\frac{E_j}{j!2^j}(\ln a)^j = \frac{2e^{\frac{\ln a}{2}}}{e^{\ln a}+1}\sum_{n=0}^{\infty}\frac{(\ln a)^n}{n!}\\ &= \frac{2a^z}{a+1}.\end{aligned}$$

Example 5.4. *The trigonometric functions $\sin(z)$ and $\cos(z)$ are entirely analytic differenceable (see Theorem 4.1). To obtain the analytic difference function of $\sin(z) = \sum_{n=0}^{\infty}\frac{(-1)^{[\frac{n}{2}]}\epsilon_n}{n!}z^n$ where $\epsilon_n = \begin{cases} 0 & ;\ n \text{ is even} \\ 1 & ;\ n \text{ is odd}\end{cases}$ first we calculate the inner series of (22) as follows*

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{(j+n)!}{j!}\frac{E_j}{2^j}c_{j+n}(z-\frac{1}{2})^n &= \sum_{j=0}^{\infty}\frac{(j+n)!}{j!}\frac{E_j}{2^j}\frac{(-1)^{[\frac{n+j}{2}]}\epsilon_{j+n}}{(n+j)!}\\ &= (-1)^{\frac{n-1}{2}}\sum_{k=0}^{\infty}(-1)^k\frac{E_{2k}}{(2k)!2^{(2k)}} \quad ; \quad n\in 2\mathbb{Z}-1.\end{aligned}$$

By using the identity $\sum_{k=0}^{\infty}(-1)^k\frac{E_{2k}}{(2k)!2^{(2k)}}=\frac{1}{\cos(\frac{1}{2})}$ *we have*

$$\sin_\delta(z)=\sum_{n=0}^{\infty}\frac{(-1)^{[\frac{n}{2}]}\epsilon_n}{n!}E_n(z)=\frac{1}{\cos(\frac{1}{2})}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(z-\frac{1}{2})^{2k+1}=\frac{\sin(z-\frac{1}{2})}{\cos(\frac{1}{2})}.$$

Also, by similar method we have

$$\cos_\delta(z)=\frac{\cos(z-\frac{1}{2})}{\cos(\frac{1}{2})}.$$

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