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Certain dense subalgebras of continuous vector-valued operator algebras

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ABSTRACT. Let X be a compact metric space with at least two elements, B be a unital commutative Banach algebra over the scalar field $\mathbb{F}(=\mathbb{R}$or $\mathbb{C})$, and $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$. Suppose that $C(X, B)$ be the continuous, $A(X, B)$ be the analytic, and $\mathrm{Lip}_\alpha(X, B)$ be the α-Lipschitz B-valued operator algebras on X. In this paper, we prove that the algebras $\mathrm{Lip}_\alpha(X, B)$ and $A(X, B)$ are dense in $C(X, B)$ under sup-norm. Also, we study the relationship between elements of the algebras $\mathrm{Lip}_\alpha(X, B)$ and $A(X, B)$.

1. Introduction

A function f from a non-empty compact metric space (X, d) into the scalar field $\mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$ is called a Lipschitz function if there exists a constant M such that the following condition hold:

$$|f(x) - f(y)| \leq Md(x, y), \qquad \forall x, y \in X.$$

The space $\mathrm{Lip}(X)$ consisting of all Lipschitz functions from X into $\mathbb{F}$ has been proved to be a Banach space, which has a series of interesting and important properties. Sherbert [7], Cao et al [2], Alimohammadi et al [1], Deville et al [4], Kupavskii et al [5] studied the Abelian Banach algebra consisting of complex-valued Lipschitz functions on a compact metric space, called the big and little Lipschitz algebra. Also Constantini studied the density of the space of continuous functions [3].

Let (X, d) be a compact metric space with at least two elements and $(B, \|\cdot\|)$ be a unital commutative Banach algebra over the scaler field $\mathbb{F}(=\mathbb{R}$or $\mathbb{C})$. Suppose that $C(X, B)$ denotes the uniform algebra of all continuous B-valued operators from X into B with the sup-norm

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|, \qquad f \in C(X, B).$$

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For each $\lambda \in \mathbb{F}$ and $f, g \in C(X, B)$ define

$$(f+g)(x) := f(x) + g(x), \quad (\lambda f)(x) := \lambda f(x), \qquad \forall x \in X.$$

It is easy to see that $(C(X, B), \| \cdot \|_\infty)$ becomes a Banach algebra over $\mathbb{F}$.
For any $f : X \to B$, set

$$L_f^\alpha(x, y) := \frac{\|f(x) - f(y)\|}{d^\alpha(x, y)}, \qquad \forall x, y \in X, \ x \neq y,$$

and

$$P_\alpha(f) := \sup_{x \neq y} L_f^\alpha(x, y),$$

which is called the Lipschitz constant of f. For $0 < \alpha \leq 1$, define

$$\mathrm{Lip}_\alpha(X, B) := \{f : X \to B : \ P_\alpha(f) < \infty\},$$

and for $0 < \alpha < 1$, define

$$\mathrm{lip}_\alpha(X, B) := \left\{ f : X \to B : \lim_{d(x,y) \to 0} L_f^\alpha(x, y) = 0 \right\}.$$

The elelemts of $\mathrm{Lip}_\alpha(X, B)$ and $\mathrm{lip}_\alpha(X, B)$ are called big and little $\alpha-$Lipschitz B-valued operators, respectively. For any $f \in \mathrm{Lip}_\alpha(X, B)$ and $\alpha \in (0, 1]$ define

$$\|f\|_\alpha := P_\alpha(f) + \|f\|_\infty.$$

In [8], the certain properties of Banach algebra $(\mathrm{Lip}_\alpha(X, B), \| \cdot \|_\alpha)$ has discussed. By a multiplicative functional on B wee shall mean a nonzero homomorphism from B to $\mathbb{C}$. The set of all multiplicative functionals on B is called the *spectrum* of B; we denote it by $\sigma(B)$.

The continuous B-valued operator f in the interior of X is called analytic when $\Lambda o f$ in the interior of X is in the usual sense analytic, where $\Lambda \in \sigma(B)$. When $B = \mathbb{F}$, put $\Lambda = I$ the identity map. We denote the set of such operators with the symbol $A(X, B)$. So

$$A(X, B) = \{f \in C(X, B) : \ \Lambda o f \text{ is analytic in the interior of } X, \Lambda \in \sigma(B)\}.$$

In this paper, we prove that the algebras $\mathrm{Lip}_\alpha(X, B)$ and $A(X, B)$ are dense subalgebras of $C(X, B)$ with sup-norm. Also, we study the relationship between elements of the algebras $\mathrm{Lip}_\alpha(X, B)$ and $A(X, B)$.

2. Preliminaries

Throughout this paper, let (X, d) be a compact metric space with at least two elements, $(B, \|.\|)$ be a unital commutative Banach algebra over the scaler filed $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) with unite e, and $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$.

According to the definitions mentioned in the introduction, for any $f \in \mathrm{Lip}_\alpha(X, B)$ we have $P_\alpha(f) < +\infty$. So, for every $x, y \in X$ we can write

$$\|f(x) - f(y)\| \leq P_\alpha(f) \ d^\alpha(x, y).$$

Then, it is easy to see that f is continuous on X. Thus, $f \in C(X, B)$. Therefore, $\mathrm{Lip}_\alpha(X, B) \subseteq C(X, B)$.

Remark 2.1. *It is obvious that for any $x, y \in X$, we see that $d(x, y) \geq 0$. So, $k_\alpha := \sup_{x,y\in X} d^\alpha(x, y)$ is a positive constant. Now let $f \in lip_\alpha(X, B)$ be arbitrary. Then $\lim_{d(x,y)\to 0} L_f^\alpha(x, y) = 0$. Thus for every $\varepsilon > 0$, there is a $\delta > 0$ such that $L_f^\alpha(x, y) < \frac{\varepsilon}{k_\alpha}$ whenever $0 < d(x, y) < \delta$. So, whenever $0 < d(x, y) < \delta$, we have*

$$\frac{\|f(x) - f(y)\|}{d^\alpha(x, y)} < \frac{\varepsilon}{k_\alpha} \quad \Rightarrow \quad \|f(x) - f(y)\| < \frac{\varepsilon}{k_\alpha} d^\alpha(x, y) < \varepsilon.$$

This shows that $f \in C(X, B)$. Therefore, $\mathrm{lip}_\alpha(X, B) \subseteq C(X, B)$.

Remark 2.2. *Let $f \in lip_\alpha(X, B)$ be arbitrary. Then there exists a $\delta > 0$ such that when $d(x, y) < \delta$, we have $\|f(x) - f(y)\| \leq d^\alpha(x, y)$ for each $x, y \in X$, i.e. in this case $P_\alpha(f) < +\infty$. Also, if put $X_\delta := \{(x, y) \in X \times X : \ d(x, y) \geq \delta\}$, then X_δ is a closed subspace of the compact space $X \times X$ and so it is compact. Define*

$$F_f : X_\delta \to B,$$
$$F_f(x, y) := \frac{f(x) - f(y)}{d^\alpha(x, y)}.$$

According to the Remark 2.1, the map F_f is continuous. Then $\sup_{(x,y)\in X_\delta} \|F_f(x, y)\| < +\infty$. Thus $\sup_{(x,y)\in X_\delta} L_f^\alpha(x, y) < +\infty$, and so $P_\alpha(f) < +\infty$ for every $x, y \in X$ whenever $d(x, y) \geq \delta$. Therefore in any case, we have $P_\alpha(f) < +\infty$ on X. So, $f \in Lip_\alpha(X, B)$. This implies that $\mathrm{lip}_\alpha(X, B) \subseteq \mathrm{Lip}_\alpha(X, B)$.

Now, let $X = [-1, 1]$. The operator f defined by $f(x) = x^2 e$ on X is a continuous operator and Lipschitz operator with Lipschitz constant 2, so $f \in \mathrm{Lip}_\alpha(X, B) \neq \emptyset$. Also for every $\alpha \in (0, 1)$, and any $x, y \in X$ with $x \neq y$, we have

$$\begin{aligned}
\lim_{d(x,y)\to 0} L_f^\alpha(x, y) &= \lim_{d(x,y)\to 0} \frac{\|f(x) - f(y)\|}{d^\alpha(x, y)} \\
&= \lim_{|x-y|\to 0} \frac{\|x^2 e - y^2 e\|}{|x - y|^\alpha} \ (\|e\| = 1) \\
&= \lim_{|x-y|\to 0} |x - y|^{1-\alpha}|x + y| = 0.
\end{aligned}$$

Then $f \in \mathrm{lip}_\alpha(X, B)$, and so $\mathrm{lip}_\alpha(X, B) \neq \emptyset$.

As well as the operator g defined by $g(x) = \sqrt[3]{x}\, e$ on $X = [-1, 1]$ is continuous, and is not Lipschitz, because for $x = \delta$, $y = -\delta$ and $\alpha = 1$, we have

$$\begin{aligned} P_\alpha(f) &= \sup_{x \neq y} L_f^\alpha(x, y) \\ &= \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d^\alpha(x, y)} \\ &= \sup_{x \neq y} \frac{\left\|\sqrt[3]{x}e - \sqrt[3]{y}e\right\|}{|x - y|} \ (\|e\| = 1) \\ &= \frac{\left|\sqrt[3]{\delta} - \sqrt[3]{-\delta}\right|}{|\delta - (-\delta)|} \\ &= \frac{2\sqrt[3]{\delta}}{2\delta} \\ &= \frac{1}{\sqrt[3]{\delta^2}} \to \infty \ as \ \delta \to 0. \end{aligned}$$

This will work together with Remarks 2.1 and 2.2:

$$\phi \neq \mathrm{lip}_\alpha(X, B) \subsetneqq \mathrm{Lip}_\alpha(X, B) \subsetneqq C(X, B).$$

Theorem 2.3. $\mathrm{lip}_\alpha(X, B)$ *is a closed subalgebra of* $\mathrm{Lip}_\alpha(X, B)$.

Proof. It is obvious that $\mathrm{lip}_\alpha(X, B)$ is a subalgebra of $\mathrm{Lip}_\alpha(X, B)$. So it is enough to prove that $\mathrm{lip}_\alpha(X, B)$ is closed. Let f be a limit point of $\mathrm{lip}_\alpha(X, B)$. Then there is a sequence $\{f_n\} \subset \mathrm{lip}_\alpha(X, B)$ such that $f_n \to f$ with $\|\cdot\|_\alpha$. So $\lim\limits_{n\to\infty} \|f_n - f\|_\alpha = 0$. Let $\epsilon > 0$. Then there is $N \in \mathbb{N}$ such that for every $n \geq N$ we have $\|f_n - f_N\|_\alpha < \frac{\epsilon}{2}$. Since $f_N \in \mathrm{lip}_\alpha(X, B)$,

$$\lim_{d(x,y)\to 0} \frac{\|f_N(x) - f_N(y)\|}{d^\alpha(x, y)} = 0, \quad (x, y \in X,\ x \neq y).$$

So there is $\delta > 0$ such that for every $t, s \in X$ with $0 < d(t, s) < \delta$ we have

$$\frac{\|f_N(t) - f_N(s)\|}{d^\alpha(t, s)} < \frac{\epsilon}{2}.$$

Thus for all $t, s \in X$ with $0 < d(t,s) < \delta$ and $n \geq N$ we have

$$\begin{aligned} \frac{\|f(t)-f(s)\|}{d^\alpha(t,s)} &= \lim_{n\to\infty} \frac{\|f_n(t)-f_n(s)\|}{d^\alpha(t,s)} \\ &= \lim_{n\to\infty} \frac{\|[(f_n-f_N)(t)-(f_n-f_N)(s)]+(f_N(t)-f_N(s))\|}{d^\alpha(t,s)} \\ &\leq \lim_{n\to\infty} P_\alpha(f_n-f_N)+\frac{\epsilon}{2} \\ &< \lim_{n\to\infty} \|f_n-f_N\|_\alpha+\frac{\epsilon}{2} \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon. \end{aligned}$$

Hence $f \in \mathrm{lip}_\alpha(X,B)$, and this proves that $\mathrm{lip}_\alpha(X,B)$ is closed. □

When $B = \mathbb{F}$, we write $C(X)$ and $\mathrm{Lip}_\alpha(X)$ instead of $C(X,B)$ and $\mathrm{Lip}_\alpha(X,B)$, respectively.

Lemma 2.4. *The algebra $\mathrm{Lip}_\alpha(X)$ is dense in $C(X)$ with sup-norm.*

PROOF. See [9]. □

Theorem 2.5. *Suppose that*

(1) *A is a closed subalgebra of $C(X)$.*
(2) *A is self-adjoint (i.e., $\bar{f} \in A$, for all $f \in A$, where the bar denotes complex conjugation.)*
(3) *A separates points on X.*
(4) *at every $x \in X$, $f(x) \neq 0$ for some $f \in A$.*

Then $A = C(X)$.

PROOF. See [6]. □

Corollary 2.6. *By Theorem 2.5, we have $\overline{A(X)} = C(X)$, i.e., the algebra $A(X)$ is dense in $C(X)$ with sup-norm, where $\overline{A(X)}$ is the closure of $A(X)$ and $A(X) = A(X,\mathbb{F})$.*

3. Main Results

In this section, we review the main results of the paper.

Theorem 3.1. *The algebra $\mathrm{Lip}_\alpha(X,B)$ is dense in $C(X,B)$ with sup-norm.*

PROOF. Let $\epsilon > 0$ and $f \in C(X,B)$ be arbitrary. We show that there exists $h \in \mathrm{Lip}_\alpha(X,B)$ such that $\|h-f\|_\infty < \epsilon$. Since $f \in C(X,B)$, $\theta \circ f \in C(X)$ for all $\theta \in \sigma(B)$. So, by Lemma 2.4, there exists $g \in \mathrm{Lip}_\alpha(X)$ such that $\|g-\theta\circ f\|_\infty < \epsilon$. Define

$$\eta : \mathbb{C} \to B,$$
$$\eta(\lambda) := \lambda e.$$

Since g is continuous, $\eta \circ g$ is continuous. Also

$$\begin{aligned}P_\alpha(\eta \circ g) =& \sup_{x \neq y} L^\alpha_{\eta\circ g}(x,y)\\ =& \sup_{x\neq y} \frac{\|(\eta\circ g)(x) - (\eta\circ g)(y)\|}{d^\alpha(x,y)}\\ =& \sup_{x\neq y} \frac{\|g(x)e - g(y)e\|}{d^\alpha(x,y)} \qquad (\|e\| = 1)\\ =& P_\alpha(g) < \infty.\end{aligned}$$

So $\eta \circ g \in \mathrm{Lip}_\alpha(X, B)$. Set $h := \eta \circ g$. Now we show that $\|h - f\|_\infty < \epsilon$. For all $x \in X$ and all $\theta \in \sigma(B)$ we have

$$|\theta\,(g(x)e - f(x))| = |g(x) - (\theta \circ f)(x)| \leq \|g - \theta \circ f\|_\infty < \epsilon, \quad (\theta(e) = 1).$$

This implies that

$$|\theta\,(\eta(g(x)) - f(x))| < \epsilon, \qquad x \in X.$$

Therefore

$$|\theta\,(\eta \circ g - f)\,(x)| < \epsilon, \qquad x \in X.$$

Since $\theta \in \sigma(B)$ is arbitrary, $\|(\eta \circ g - f)(x)\| < \epsilon$, $(x \in X)$. Consequently, $\|\eta \circ g - f\|_\infty < \epsilon$ or $\|h - f\|_\infty < \epsilon$. This completes the proof. □

Theorem 3.2. *The algebra $A(X, B)$ is dense in $C(X, B)$ with sup-norm.*

PROOF. Let $f \in C(X, B)$ and $\varepsilon > 0$ be arbitrary. We show that there is $g \in A(X, B)$ such that $\|f - g\|_\infty < \varepsilon$. Since $f \in C(X, B)$, $\Lambda \circ f \in C(X)$ for every $\Lambda \in \sigma(B)$. By Corollary 2.6, there exists $h \in A(X)$ such that $\|\Lambda \circ f - h\|_\infty < \varepsilon$. So, we have

$$\begin{aligned}&\sup_{x\in X} |(\Lambda \circ f - h)(x)| < \varepsilon,\\ \Rightarrow &\sup_{x\in X} |\Lambda(f(x)) - h(x)| < \varepsilon,\\ \Rightarrow &\sup_{x\in X} |\Lambda(f(x) - h(x)e)| < \varepsilon, \quad (\Lambda(e) = 1).\end{aligned}$$

Since $\Lambda \in \sigma(B)$ is arbitrary,

$$\begin{aligned}&\sup_{x\in X} \|f(x) - h(x)e\| < \varepsilon,\\ \Rightarrow &\sup_{x\in X} \|(f - he)(x)\| < \varepsilon,\\ \Rightarrow &\|f - he\|_\infty < \varepsilon.\end{aligned}$$

Take $g := he$, then it is obvious that $g \in A(X, B)$ and $\|f - g\|_\infty < \varepsilon$. □

Corollary 3.3. *Each element of $A(X, B)$ can be approximated by elements of $\mathrm{Lip}_\alpha(X, B)$ with sup-norm.*

Proof. Let $f \in A(X, B)$ be arbitrary. Since $f \in A(X, B)$, $f \in C(X, B)$. So by Theorem 3.1, there is $g \in \mathrm{Lip}_\alpha(X, B)$ such that $\|f - g\|_\infty < \varepsilon$ for every $\varepsilon > 0$. This completes the proof. □

Corollary 3.4. *By using Theorem 3.2, each element of $Lip_\alpha(X, B)$ can be approximated by elements of $A(X, B)$ with sup-norm.*

Corollary 3.5. *Any continuous operator $f \in C(X, B)$ can be approximated by elements of $Lip_\alpha(X, B)$ and $A(X, B)$ with at most difference $\varepsilon > 0$ under sup-norm. This means that for any $f \in C(X, B)$ and $\varepsilon > 0$,*

- *since $\mathrm{Lip}_\alpha(X, B)$ is dense in $C(X, B)$ by Theorem 3.1, there exists g in $\mathrm{Lip}_\alpha(X, B)$ such that $\|f - g\|_\infty < \frac{\varepsilon}{2}$,*
- *since $A(X, B)$ is dense in $C(X, B)$ by Theorem 3.2, there exists $h \in A(X, B)$ such that $\|h - f\|_\infty < \frac{\varepsilon}{2}$.*

Hence

$$\begin{aligned} \|h - g\|_\infty &= \|(h - f) + (f - g)\|_\infty \\ &\le \|h - f\|_\infty + \|f - g\|_\infty \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

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