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# Jafari variational iteration method for solving one-dimensional fractional diffusion equations

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Abstract. H. Jafari proposed a new integral transform recently, namely, the Jafari transform, which covered all classes of integral transforms in the class of Laplace transform, such as Laplace, Sumudu, Elzaki, Aboodh, natural, and Shehu transformation, etc. In this paper, we utilize a semi-analytical technique, namely the Jafari variational iteration method, abbreviated JVIM, and we apply this technique to resolve one-dimensional diffusion equations with fractional-order type using the Caputo fractional derivative. The results are compared with homotopy analysis Shehu transform method (HASTM). Also, the results show the suggested algorithm is efficient, accurate, and a powerful technique for solving a wide variety of linear and non-linear problems arising in various scientific areas.

## 1. Introduction

Mathematical models are created from assumptions inspired by the observation of some real phenomena in the hope that the model behavior resembles the real behavior. Therefore, partial differential equations are used in many areas of science and engineering to better understand them. Unfortunately, an accurate solution to these problems is very difficult, especially for nonlinear problems. Therefore, in

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the last few decades, researchers have sought to find accurate methods for solving nonlinear differential equations; see [11, 38, 20, 31, 32, 23, 3].

Historically, the notion of fractional calculus is very old; it was first introduced by Leibniz and L'Hopital in the year 1695 [37]. Since then, the concept of fractional calculus has been used in many real-life problems since it has properties to explain and make predictions about natural phenomena more accurately than classical calculus [6]. Fractional diffusion equations gained considerable popularity in synchronization, mechanical systems, control, plasma physics, quantum mechanics, chaos, and a dynamic system. In the literature, there are different methods utilized for solving the fractional diffusion equations. Such as the homotopy analysis Shehu transform method (HASTM) [33]. The Homotopy analysis method (HAM) [3, 19]. The Adomian decomposition method (ADM) [17]. The variational iteration method (VIM) [12, 14, 15, 35] . The homotopy perturbation method (HPM) [13, 18]. The homotopy analysis Sumudu transform method (HASTM) [29]. The reduce differential transform method (RDTM) [41] and others. Among these the variational iteration method (VIM), proposed by J.H. He [12, 14, 15, 35], which was successfully applied for solving linear and non-linear problems. On the other hand, during the last two decades, many researchers have introduced integral transforms in the class of Laplace transforms, such as Sumudu, Elzaki, Natural, Aboodh, and Shehu transform, etc., see[25, 40, 42] and [2, 4, 36, 30, 9, 10, 22, 27, 28]. Therefore, H. Jafari introduced a generalized integral transform that covered all classes of integral transforms in the class of Laplace transform. This transform was used for solving ODEs, integral equations, and fractional integral equations [34, 16].

In this work, we study the one-dimensional diffusion equations with fractional-order type under the Caputo fractional derivative. The proposed technique is a combination of the Jafari transform and the variational iteration method. The method is called the Jafari variational iteration method, abbreviated JVIM. Further, the homotopy analysis Shehu transform method (HASTM) [33] is compared with the proposed method.

The structure of this paper is organized as follows: In Section (2), we present the Jafari transform and some definitions of fractional calculus. In Section (3), the JVIM and relation between other transforms are analyzed. In Section (4), the approximate solution for the one-dimensional diffusion equations is presented. Finally, in Section (5), some conclusions are presented.

## 2. Jafari Transform and Fractional Calculus

Some basic definitions of fractional calculus and the Jafari transform are used later in this paper.

**Definition 2.1.** The Riemann-Liouville fractional integral of order $\alpha \geq 0$, of a function $f(t) : (0, +\infty) \to \mathbb{R}$, is defined as [26, 24].

$$\begin{cases} I^{\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ I^{\alpha} f(t) = f(t), & \alpha = 0. \end{cases} \tag{1}$$

where $\Gamma(.)$ is the gamma function.

**Definition 2.2.** The Caputo fractional derivative of order $n-1 < \alpha < n$, of function $f(t) : (0, +\infty) \to \mathbb{R}$, is defined as [26, 24].

$${}^{c}D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & \alpha \neq n \in \mathbb{R} - \mathbb{N}, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}. \end{cases} \tag{2}$$

In particular,

$${}^{c}D^{\alpha} t^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} t^{m-\alpha}.$$

**Definition 2.3.** The generalized Mittag-Leffler function (two parameters) is defined as: [34, 24]

$$E_{\alpha,\beta}(at) = \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(ak+\beta)}, \quad \alpha > 0, \alpha, \beta \in \mathbb{R}, t \in \mathbb{C}. \tag{3}$$

In particular, if $\beta = 1$, we have (one parameter)

$$E_{\alpha}(at) = \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(ak+1)}. \tag{4}$$

**Definition 2.4.** Let $f(t)$ be an integrable function defied for $t \geq 0, p(s) \neq 0$, and $q(s)$ are positive real functions, we define the Jafari transform of $f(t)$, denoted by $T[f(t)]$, by the formula [16]

$$T[f(t)] = p(s) \int_0^{\infty} f(t) e^{-q(s)t} dt = \widetilde{f}(s), \tag{5}$$

provided the integral exists for some $q(s)$.

**Definition 2.5.** If $n \in \mathbb{Z}^+$, where $n-1 < \alpha \leq n$ and $\widetilde{f}(s)$ be the Jafari transform of the function $f(t)$, then the Jafari transform of the Caputo fractional derivative of order $\alpha > 0$, is[16, 24]

$$T[{}^{c}D^{\alpha}(f(t))] = q^{\alpha}(s) \widetilde{f}(s) - p(s) \sum_{i=0}^{n-1} q^{\alpha-1-i} f^{(i)}(0). \tag{6}$$

## 3. Jafari Variational Iteration Method (JVIM)

In this section, we discuss the JVIM solution to the fractional partial differential equations. Let us consider the following nonlinear fractional differential equation

$$^{c}D^{\alpha}\nu(x,t) + L\nu(x,t) + \aleph\nu(x,t) = g(x,t), \quad x > 0. \tag{7}$$

subject to the initial condition

$$\nu(x,0) = f(x). \tag{8}$$

where $L$ is a linear operator, $\aleph$ represents a nonlinear operator, $g(x,t)$ is the source term, and $^{c}D^{\alpha}(.)$ is the Caputo fractional derivative of order $\alpha$ where $0 < \alpha \leq 1$. The solution of an algebraic equation $f(x) = 0$, by using the Lagrange multipliers is given by

$$x_{n+1} = x_n + \lambda f(x_n). \tag{9}$$

The optimality condition for the extreme $\dfrac{\delta x_{n+1}}{\delta x_n}$ Leads to

$$\lambda = -\frac{1}{f'(x_n)}, \tag{10}$$

From (9) and (10), we have the approximate solution

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots \tag{11}$$

Now, takeing the Jafari transform to (7), we have an algebraic equation as follows:

$$q^{\alpha}(s)\widetilde{\nu}(x,s) - p(s)q^{\alpha-1}(s)\nu(x,0) + T[L\nu(x,t) + \aleph\nu(x,t) - g(x,t)] = 0, \tag{12}$$

where $T\nu(x,t) = \widetilde{\nu}(x,s)$, the iteration formula of (12), by using (11), given by

$$\widetilde{\nu}_{n+1}(x,s) = \widetilde{\nu}_n(x,s) + \lambda(s)(q^{\alpha}(s)\widetilde{\nu}(x,s) - p(s)q^{\alpha-1}(s)\nu(x,0)), \tag{13}$$

Let us assume that $T[L\nu(t) + \aleph\nu(t) - g(t)]$ is a restricted terms. We can derive a Lagrange multiplier by taking the variation of (13), as:

$$\begin{aligned}
\delta\widetilde{\nu}_{n+1}(x,s) =& \delta\widetilde{\nu}_n(x,s) + \delta\lambda(s)(q^{\alpha}(s)\widetilde{\nu}(x,s)) \\
=& \delta\widetilde{\nu}_n(x,s) + \lambda(s)q^{\alpha}(s)(\delta\widetilde{\nu}(x,s)) \\
=& 0.
\end{aligned}$$

Therefore, the Lagrange multiplier, can be identified as

$$\lambda(s) = -\frac{1}{q^{\alpha}(s)}. \tag{14}$$

As a result, we obtain the following iteration formula after takeing the inverse Jafari transform:

$$\nu_{n+1}(x,t) = \nu_n(x,0) - T^{-1}\Big[\frac{1}{q^{\alpha}(s)}[T[L\nu_n(x,t) + \aleph\nu_n(x,t) - g(x,t)]]\Big], \tag{15}$$

Consequently, an approximate solution may be procured using

$$\nu(x,t) = \lim_{n\to\infty} \nu_n(x,t) \tag{16}$$

**Corollary 3.1.** *In view of (14), we have:*

- *If $p(s) = 1$ and $q(s) = s$, then the Lagrange multiplier, by using the Laplace transform, is given by $\lambda(s) = -\frac{1}{s^\alpha}$, see [5, 43].*
- *If $p(s) = \frac{1}{s}$ and $q(s) = \frac{1}{s}$, then the Lagrange multiplier, by using the Sumudu transform, is given by $\lambda(s) = -s^\alpha$, see [21].*
- *If $p(s) = s$ and $q(s) = \frac{1}{s}$, then the Lagrange multiplier, by using Elzaki transform, is given by $\lambda(s) = -s^\alpha$, see [44].*
- *If $p(s) = \frac{1}{s}$ and $q(s) = s$, then the Lagrange multiplier, by using the Aboodh transform, is given by $\lambda(s) = -\frac{1}{s^\alpha}$, see [8].*
- *If $p(s) = \frac{1}{\nu}$ and $q(s) = \frac{s}{\nu}$, then the Lagrange multiplier, by using the natural transform, is given by $\lambda(s) = -\left(\frac{\nu}{s}\right)^\alpha$, see [1].*
- *If $p(s) = 1$ and $q(s) = \frac{s}{\nu}$, then the Lagrange multiplier, by using the Shehu transform, is given by $\lambda(s) = -\left(\frac{\nu}{s}\right)^\alpha$, see [39, 7].*

## 4. Applications of the JVIM

In this section, the JVIM is efficiently applied to the fractional diffusion equation to validate its efficiency and high accuracy.

**Example 4.1.** Consider the following one-dimensional linear fractional diffusion equation [33]:

$$\frac{\partial^\alpha \nu(x,t)}{\partial t^\alpha} = \frac{\partial^2 \nu(x,t)}{\partial x^2} + \nu(x,t), \tag{17}$$

subject to the initial condition

$$\nu(x,0) = \cos(\pi x), \quad 0 \le x \le 1. \tag{18}$$

In view of (15), the iteration formula of equation (17) is given by

$$\nu_{n+1}(x,t) = \nu_n(x,0) - T^{-1}\Big[\frac{1}{q^\alpha(s)}\Big[T\Big[\frac{\partial^\alpha \nu_n(x,t)}{\partial t^\alpha} - \frac{\partial^2 \nu_n(x,t)}{\partial x^2} - \nu_n(x,t)\Big]\Big]\Big], \tag{19}$$

Consequently, beginning with $\nu_0(x,0)=\nu(x,0)=\cos(\pi x)$, we find the following approximations

$$\begin{aligned}
\nu_1(x,t) =&\nu_0(x,0) - T^{-1}\Big[\frac{1}{q^{\alpha}(s)}\Big[T\Big[\frac{\partial^{\alpha}\nu_0(x,t)}{\partial t^{\alpha}} - \frac{\partial^2\nu_0(x,t)}{\partial x^2} - \nu_0(x,t)\Big]\Big]\Big]\\
=&\cos(\pi x) + T^{-1}\Big[\frac{1}{q^{\alpha}(s)}\Big[T\big[(1-\pi^2)\cos(\pi x)\big]\Big]\Big]\\
=&\cos(\pi x) + T^{-1}\Big[\frac{1}{q^{\alpha}(s)}\Big[T(1-\pi^2)\cos(\pi x)\frac{p(s)}{q(s)}\Big]\Big]\\
=&\cos(\pi x) + (1-\pi^2)\cos(\pi x)\frac{t^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}$$

$$\begin{aligned}
\nu_2(x,t) =&\nu_1(x,0) - T^{-1}\Big[\frac{1}{q^{\alpha}(s)}\Big[T\Big[\frac{\partial^{\alpha}\nu_1(x,t)}{\partial t^{\alpha}} - \frac{\partial^2\nu_1(x,t)}{\partial x^2} - \nu_1(x,t)\Big]\Big]\Big]\\
=&\cos(\pi x) + (1-\pi^2)\cos(\pi x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (1-\pi^2)^2\cos(\pi x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
\nu_3(x,t) =&\nu_2(x,0) - T^{-1}\Big[\frac{1}{q^{\alpha}(s)}\Big[T\Big[\frac{\partial^{\alpha}\nu_2(x,t)}{\partial t^{\alpha}} - \frac{\partial^2\nu_2(x,t)}{\partial x^2} - \nu_2(x,t)\Big]\Big]\Big]\\
=&\cos(\pi x) + (1-\pi^2)\cos(\pi x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (1-\pi^2)^2\cos(\pi x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&+ (1-\pi^2)^3\cos(\pi x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},
\end{aligned}$$

and so on. Thus, we have

$$\nu_n(x,t) = \sum_{m=0}^{n}(1-\pi^2)^m\cos(\pi x)\frac{t^{m\alpha}}{\Gamma(m\alpha+1)},$$

so that the solution $\nu(x,t)$ of the equation (17) is given by

$$\begin{aligned}
\nu(x,t) &= \lim_{n\to\infty}\nu_n(x,t) = \nu_n(x,t)\\
&= \sum_{m=0}^{n}(1-\pi^2)^m\cos(\pi x)\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\\
&= \cos(\pi x)E_{\alpha}[(1-\pi^2)t^{\alpha}].
\end{aligned}\tag{20}$$

The result is the same as HASTM [33].

**Example 4.2.** Consider the following nonlinear fractional wave equation [33]:

$$\frac{\partial^{\alpha}\nu(x,t)}{\partial t^{\alpha}} = \frac{\partial^2\nu(x,t)}{\partial x^2} - \frac{1}{2}\frac{\partial\nu^2(x,t)}{\partial x}, \qquad 0<\alpha\le 1, \tag{21}$$

subject to the initial condition

$$\nu(x,0) = x. \tag{22}$$

In view of equation (15), we have

$$\nu_{n+1}(x,t)=\nu_n(x,0)-T^{-1}\Big[\frac{1}{q^{\alpha}(s)}\Big[T\Big[\frac{\partial^{\alpha}\nu_n(x,t)}{\partial t^{\alpha}}-\frac{\partial^2\nu_n(x,t)}{\partial x^2}+\frac{1}{2}\frac{\partial\nu_n^2(x,t)}{\partial x}\Big]\Big]\Big]. \quad (23)$$

Consequently, beginning with $\nu_0(x,0)=\nu(x,0)=x$, we find the following approximations

$$\begin{aligned}\nu_1(x,t)=&\nu_0(x,0)-T^{-1}\Big[\frac{1}{q^{\alpha}(s)}\Big[T\Big[\frac{\partial^{\alpha}\nu_0(x,t)}{\partial t^{\alpha}}-\frac{\partial^2\nu_0(x,t)}{\partial x^2}+\frac{1}{2}\frac{\partial\nu_0^2(x,t)}{\partial x}\Big]\Big]\Big]\\ =&x-T^{-1}\Big[\frac{1}{q^{\alpha}(s)}[T[x]]\Big]=x-T^{-1}\Big[\frac{p(s)}{q^{\alpha+1}(s)}x\Big]=x-x\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\ \nu_2(x,t)=&x-x\frac{t^{\alpha}}{\Gamma(\alpha+1)}+2x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\ \nu_3(x,t)=&x-x\frac{t^{\alpha}}{\Gamma(\alpha+1)}+2x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-x\Big(\frac{1}{\Gamma^2(\alpha+1)}+\frac{4}{\Gamma^2(2\alpha+1)}\Big)\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)},\end{aligned}$$

and so on. In particular, when $\alpha\to 1$, we obtain an exact solution

$$\nu(x,t)\cong x(1-t+t^2-t^3)=\frac{x}{t+1}. \quad (24)$$

which is the same as given by HASTM [33].

## 5. Conclusion

In this study, the JVIM has been successfully applied to obtain the solutions of the one-dimensional fractional diffusion equations. In view of the results, the relationship between the proposed method and the combination of the variational iteration method with integral transforms in the class of Laplace transform is proved. Moreover, we can say that this technique is a powerful mathematical tool for solving FPDEs. In the future, we will develop the proposed method under other fractional operators, such as non-local and non-singular fractional derivatives.

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