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Some common fixed points results via the degree of nondensifiability

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Abstract. In this paper, under suitable conditions and by using the so-called degree of nondensifiability (DND), we provide sufficient conditions for the existence of a common fixed point for two commuting self-mappings defined into a non-empty, bounded, closed and convex subset of a Banach space. Our main result generalizes a Darbo-type fixed point theorem based on the DND. To illustrate the differences between our results and a known common fixed point result for two commuting self-mappings due to Jungck or others based on the measures of noncompactness, we provide some examples.

1. Introduction

It is a well known fact that the celebrated Banach fixed point theorem [**4**], *inter alia* due to its many applications, has been widely generalized in many directions (see, for instance, [**17, 20, 21**] and references therein). In this paper, we are interested in the generalization proved by Jungck [**18**] in 1976, which we recall below.

In what follows, (Y, d) will be a complete metric space. To simplify writing, we give the following definition:

Definition 1.1. Let $T, S : (Y, d) \longrightarrow (Y, d)$ and $\lambda \in (0, 1)$. We will say that T is a (S, λ)-contractive mapping if the following conditions hold:

(i) S and T commute, that is, $T(S(x)) = S(T(x))$ for all $x \in Y$.
(ii) S is continuous and $T(Y) \subset S(Y)$.
(iii) $d(T(x), T(y)) \leq \lambda d(S(x), S(y))$, for all $x, y \in Y$.

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Remark 1.1. *Let us note that, from condition (iii) of the above definition, a* (S, λ)*-contractive mapping* T *is continuous.*

Now, we can state the Jungck fixed point theorem mentioned above:

Theorem 1.2. *Let* T *be a* (S, λ)*-contractive mapping for some* $S : (Y, d) \longrightarrow (Y, d)$ *and* $\lambda \in (0, 1)$*. Then,* T *and* S *have a unique common fixed point, that is to say, there is a unique* $x^* \in Y$ *such that* $T(x^*) = S(x^*) = x^*$*.*

As in the case of the Banach fixed point theorem, there are many generalizations of the above result (see, for instance, [27] and reference therein).

On the other hand, based in the *degree of nondensifiability*, shortly DND and explained in detail in Section 2, in this paper we propose a new generalization of Definition 1.1, for the case that (Y, d) be a non-empty, bounded, closed and convex subset of a Banach space, as well as some common fixed points results (see Theorem 3.3 and Corollary 3.4). Roughly speaking, the DND is defined from the so called α*-dense curves* and measures (in the specified sense) the distance from a non-empty and bounded subset of a Banach space to the class of its Peano continua. We recall that a Peano continua is a compact, connected and locally connected metric space or equivalently, according with the Hahn-Mazurkiewicz theorem (see, for instance, [26]), the continuous image of the closed unit interval $I := [0, 1]$.

Moreover, our main result (see also Corollary 3.4) contains, as a particular case, Theorem 2.3, which is a Darbo-type fixed point theorem based in the DND. Through several examples, we show that our results can be applied in cases where neither Theorem 1.2 or other common fixed points results based in the so called *measures of noncompactness* do not work.

To close our exposition, we point out in Section 4 some remarks related with the presented concepts and results.

2. The degree of nondensifiability

Throughout this section (X, d) will be a metric space and $\mathcal{B}(X)$ the class of the non-empty and bounded subsets of X. As usual, for a given $B \in \mathcal{B}(X)$, we denote by $\bar{B}$ and $\mathrm{Diam}(B)$ the closure and the diameter of B. In the particular case that X be a linear space, $\mathrm{Conv}(B)$ stands for the convex hull of B.

In 1997 Cherruault and Mora [23] introduced the following concepts:

Definition 2.1. Let $B \in \mathcal{B}(X)$ and $\alpha \geq 0$. A continuous mapping $\gamma : I \longrightarrow X$ is said to be an α-dense curve in B if the following conditions hold:

(i) $\gamma(I) \subset B$.
(ii) For each $x \in B$ there is $y \in \gamma(I)$ such that $d(x, y) \leq \alpha$.

If for each $\alpha > 0$ there is an α-dense curve in B, then B is said to be densifiable.

Let us note that the concept of α-dense curve is a generalization of the so called *space-filling curves*, see [26]. Also, for a given $B \in \mathcal{B}(X)$, there is always an α-dense curve in B for each $\alpha \geq \mathrm{Diam}(B)$, the diameter of B. Indeed, fixed $x_0 \in B$ the mapping $\gamma(t) := x_0$ for all $t \in I$ is trivially such α-dense curve. For a detailed exposition of the above concepts, see [7, 22, 23, 24, 25] and references therein.

On the other hand, from the α-dense curves we can define the following (see [13, 24]):

Definition 2.2. For a given $B \in \mathcal{B}(X)$ the degree of nondensifiability, in short DND, of B is defined as

$$\phi(B) := \inf\{\alpha \geq 0 : \Gamma_{B,\alpha} \neq \emptyset\},$$

$\Gamma_{B,\alpha}$ being the class of the α-dense curves in B.

Note that ϕ is well defined. Indeed, from the above considerations we have $0 \leq \phi(B) \leq \mathrm{Diam}(B)$ for all $B \in \mathcal{B}(X)$.

Example 2.3. (see [24]) Let U_X be the closed unit ball of a Banach space X. Then,

$$\phi(U_X) = \begin{cases} 1, & \text{if } X \text{ has infinite dimension} \\ \\ 0, & \text{if } X \text{ has finite dimension} \end{cases}.$$

The main properties of ϕ are stated in the following result:

Proposition 2.1. *The DND ϕ satisfies:*

(M-1) Regularity on the subfamily $\mathcal{B}_{\mathrm{arc}}(X) \subset \mathcal{B}(X)$ of arc-connected subsets of $\mathcal{B}(X)$: $\phi(B) = 0$ if, and only if, B is totally bounded, for each $B \in \mathcal{B}_{\mathrm{arc}}(X)$.

(M-2) Invariant under closure: $\phi(B) = \phi(\bar{B})$, for each $B \in \mathcal{B}(X)$.

Additionally, if X is a Banach space, then the following conditions are also satisfied:

(B-1) Semi-homogeneity: $\phi(cB) = |c|\phi(B)$, for each $c \in \mathbb{R}$ and $B \in \mathcal{B}(X)$.

(B-2) Invariant under translations: $\phi(x_0 + B) = \phi(B)$, for each $x_0 \in X$ and $B \in \mathcal{B}(X)$.

(B-3) $\phi(\mathrm{Conv}(B)) \leq \phi(B)$, for each $B \in \mathcal{B}(X)$.

(B-4) $\phi(\mathrm{Conv}(B_1 \cup B_2)) \leq \max\{\phi(\mathrm{Conv}(B_1)), \phi(\mathrm{Conv}(B_2))\}$, for each $B_1, B_2 \in \mathcal{B}(X)$.

(B-5) Generalized Cantor's intersection theorem: If $(C_n)_{n\geq 1}$ is a decreasing sequence of non-empty, closed and convex subsets of X with $\lim_n \phi(C_n) = 0$, then the intersection of all C_n is non-empty, convex and compact.

Proof. Properties (M1), (M2) and (B1)-(B4) was proved in [13] and (B-5) in the proof of Theorem 2.1 of [10]. □

From now on, $(X, \|\cdot\|)$ will be a Banach space.

Note that the properties of the DND ϕ exposed in the above result are similar to the properties of the so called *measures of noncompactness*, shortly MNCs (see, for instance, [**1**, **3**]). But as has been demonstrated in [**13**], the DND ϕ is not a MNC. Moreover, through the papers [**8**, **9**, **10**, **11**, **12**] we have proved that ϕ and the MNCs are essentially different concepts, and consequently the fixed point results based in the DND ϕ can be applied where other similar fixed points results based in the MNCs do not work (see also Example 2.4 below).

However, there are some relationships between the DND ϕ and the MNCs. For instance, if χ is the Hausdorff MNC (see [**1**, **3**]) defined as

$$\chi(B) := \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many balls with radius} \leq \varepsilon\},$$

for all $B \in \mathcal{B}(X)$, we have the following result (see [**13**]):

Proposition 2.2. *For each $B \in \mathcal{B}_{\text{arc}}(X)$, the inequalities*

$$\chi(B) \leq \phi(B) \leq 2\chi(B),$$

hold and are the best possible in infinite dimensional Banach spaces.

In [**10**, **12**], based in the DND ϕ, we have proved a similar result of the well known Darbo fixed point theorem for MNCs (again [**1**, **3**]):

Theorem 2.3. *Let $C \in \mathcal{B}(X)$ closed and convex and $T : C \longrightarrow C$ continuous. Assume that there is $\lambda \in (0, 1)$ such that*

$$\phi(T(B)) \leq \lambda\phi(B), \quad \text{for all non-empty and convex } B \subset C.$$

Then, T has some fixed point.

As we have pointed out above, ϕ is not a MNC. So, one can expect that Theorem 2.3 and Darbo fixed point theorem be different. To evidence this fact and conclude this section, we show below the example given in [**9**, Example 3.4] (see also [**12**, Example 3.2]).

Example 2.4. Let $C(I)$ be the Banach space of the continuous functions defined on I, endowed its usual supremum norm, and the set $C := \{x \in C(I) : 0 = x(0) \leq x(t) \leq 1 = x(1), t \in I\}$. Consider the mapping $T : C \longrightarrow C$ defined as

$$T(x)(t) := \begin{cases} \dfrac{1}{2}x(2t), & 0 \leq t \leq \dfrac{1}{2} \\ \\ \dfrac{1}{2}x(2t-1) + \dfrac{1}{2}, & \dfrac{1}{2} < t \leq 1 \end{cases}, \quad \text{for all } x \in C \text{ and } t \in I.$$

Then, $\chi(C) = \chi(T(C)) = \frac{1}{2}$ and therefore the conditions of Darbo fixed point theorem do not hold in this case for the MNC χ. However, as $\phi(T(B)) \leq \phi(B)/2$ for each $B \subset C$ non-empty and convex, conditions of Theorem 2.3 hold.

3. The main result

Let us introduce the following concept:

Definition 3.1. Let $T, S : C \longrightarrow C$, $C \in \mathcal{B}(X)$ closed and convex, and $\lambda \in (0,1)$. We will say that T is a DND-(S,λ)-contractive mapping if the following conditions hold:

(i) T and S commute.
(ii) T and S are continuous.
(iii) $\phi(T(B)) \leq \lambda\phi(S(B))$, for all $B \subset C \cap \mathcal{B}_{\mathrm{arc}}(X)$.

In the next result, we prove that the above concept generalizes, in the class $\mathcal{B}_{\mathrm{arc}}(X)$, the given one in Definition 1.1. Also, to show that it is a *real generalization*, we present in Example 3.2 (see also Example 3.3) a DND-(S,λ)-contractive mapping which is not a (S,λ)-contractive mapping for any $\lambda \in (0,1)$.

Proposition 3.1. *Let T be a (S,λ)-contractive mapping, for some $S : C \longrightarrow C$, with $C \in \mathcal{B}_{\mathrm{arc}}(X)$ closed and convex and $\lambda \in (0,1)$. Then, T is a DND-(S,λ)-contractive mapping.*

Proof. Property (i) of Definition 3.1 is, precisely, the given one in Definition 1.1. By Remark 1.1, T is continuous and therefore (ii) of Definition 3.1 is satisfied. We prove in the next lines property (iii).

Let $B \subset C \cap \mathcal{B}_{\mathrm{arc}}(X)$, $\alpha := \phi(S(B))$ and take any $\varepsilon > 0$. From the definition of ϕ, there is an $(\alpha + \frac{\varepsilon}{2})$-dense curve in $S(B)$, put γ. So, we have

$$S(B) \subset \gamma(I) + (\alpha + \frac{\varepsilon}{2})U_X, \tag{3.1}$$

where U_X denotes the closed unit ball of X. As $\gamma(I) \subset S(B)$ is compact, there exists a finite subset of $\gamma(I)$, put $\{S(x_1), \dots, S(x_n)\}$ for certain $x_1, \dots, x_n \in B$, such that

$$\gamma(I) \subset \{S(x_1), \dots, S(x_n)\} + (\alpha + \frac{\varepsilon}{2})U_X. \tag{3.2}$$

Therefore, from (3.1) and (3.2), we obtain the inclusion

$$S(B) \subset \{S(x_1), \dots, S(x_n)\} + (\alpha + \varepsilon)U_X. \tag{3.3}$$

Next, consider a continuous mapping $p : I \longrightarrow B$ joining the vectors $x_1, \dots, x_n$. Note that p is well defined because of $B \in \mathcal{B}_{\mathrm{arc}}(X)$. Also, define $\tilde{\gamma} := T \circ p : I \longrightarrow X$. Is clear that $\tilde{\gamma}$ is continuous and $\tilde{\gamma}(I) \subset T(B)$.

Let any $y \in T(B)$, put $y := T(x)$ for some $x \in B$. Noticing (3.3), there is $1 \leq i \leq n$ such that

$$\|S(x) - S(x_i)\| \leq \alpha + \varepsilon. \tag{3.4}$$

Then, if $t_i \in I$ is such that $p(t_i) = x_i$ and taking into account that T is a (S, λ)-contractive mapping and inequality (3.4) we have

$$\|y - \tilde{\gamma}(t_i)\| = \|T(x) - \tilde{\gamma}(t_i)\| = \|T(x) - T(x_i)\| \leq \lambda \|S(x) - S(x_i)\| \leq \lambda(\alpha + \varepsilon).$$

So, we conclude that $\tilde{\gamma}$ is a $\lambda(\alpha + \varepsilon)$-dense curve in $T(B)$, and the result follows from the arbitrariness of $\varepsilon > 0$.

□

Example 3.2. As usual, by c_0 let us denote the space of all sequences which are convergent to zero with the supremum norm. Consider two continuous functions $f, g : I \longrightarrow I$ such that

(i) $f(g(t)) = g(f(t))$ for all $t \in I$.
(ii) f and g do not have any common fixed point.

The existence of such functions was proved (independently) in [6, 15]. Let U_{c_0} be the closed unit ball of c_0 and $T, S : U_{c_0} \longrightarrow U_{c_0}$ given by

$$T(x) := (f(|x_1|), 0, \ldots, 0, \ldots), \quad S(x) := (g(|x_1|), x_2, \ldots, x_n, \ldots),$$

for all $x := (x_n)_{n \geq 1} \in U_{c_0}$. Is clear, from (i), that T and S are continuous and commute. Let us note that given $B \subset U_{c_0} \cap \mathcal{B}_{\mathrm{arc}}(X)$, in view of (M-1) of Proposition 2.1, $\phi(T(B)) = 0$ and therefore

$$\phi(T(B)) \leq \lambda \phi(S(B)), \quad \text{for all } \lambda \in (0, 1).$$

So, T is a DND-(S, λ)-contraction for all $\lambda \in (0, 1)$. However, as T and S do not have any common fixed point (by virtue of (ii)), noticing Theorem 1.2, T is not a (S, λ)-contraction for any $\lambda \in (0, 1)$.

Remark 3.2. *A particular case of* [5, Theorem 3.1] *is the following: if $T, S : C \longrightarrow C$, $C \in \mathcal{B}(X)$ closed and convex, are continuous then T and S have, at least, one common fixed point, whenever the following conditions hold:*

(a) T and S commute .
(b) For each non-empty $B \subset C$, $\mu(T(B)) \leq \lambda \mu(S(B))$ for some MNC μ and $\lambda \in (0, 1)$.

However, in view of the above example, it seems that, in general, the above result is not correct at all.

Before to continue, it is convenient recall that a mapping S, defined into a subset C of X, is said to be *affine* (see, for instance, [19, Definition 2.4]) if given any convex $M \subset C$

$$S(kx + (1-k)y) = kS(x) + (1-k)S(y), \quad \text{for all } x, y \in M \text{ and } k \in (0, 1).$$

At this point, we present our main result:

Theorem 3.3. *Let $C \in \mathcal{B}(X)$ closed and convex and $T : C \longrightarrow C$ a DND-(S,λ)-contractive mapping, for some $S : C \longrightarrow C$ and $\lambda \in (0,1)$. Assume that S is affine and*

$$\phi(S^k(\overline{\text{Conv}}(B))) \leq \phi(S^k(B)), \quad \text{for each non-empty } B \subset C \text{ and } k \geq 1. \tag{3.5}$$

Then, T and S have some common fixed point.

Proof. First, we will prove that T has some fixed point in C. Define $C_0 := C$ and $C_n := \overline{\text{Conv}}(T(C_{n-1}))$ for each $n \geq 1$. Then, by Proposition 2.1, condition (iii) of Definition 3.1 and (3.5) we have

$$\phi(C_n) = \phi(\overline{\text{Conv}}(T(C_{n-1})) \leq \phi(T(C_{n-1})) \leq \lambda\phi(S(C_{n-1})) = \lambda\phi(S(\overline{\text{Conv}}(T(C_{n-2})))) \leq$$

$$\lambda\phi(S(T(C_{n-2}))) = \lambda\phi(T(S(C_{n-2}))) \leq \lambda^2\phi(S^2(C_{n-2})) = \lambda^2\phi(S^2(\overline{\text{Conv}}(T(C_{n-3})))) \leq$$

$$\lambda^2\phi(S^2(T(C_{n-3}))) = \lambda^2\phi(T(S^2(C_{n-3}))) \leq \lambda^3\phi(S^3(C_{n-3})) \leq \ldots \leq \lambda^n\phi(S^n(C_0)) =$$

$$\lambda^n\phi(S^n(C)).$$

So, because of $S^k(C) \subset C$ for all $k \geq 1$ and the considerations of Section 2, we infer that

$$\lim_n \phi(C_n) \leq \lim_n \lambda^n\phi(S^n(C)) \leq \lim_n \lambda^n \text{Diam}(S^n(C)) \leq \lim_n \lambda^n \text{Diam}(C) = 0$$

and consequently, by (B-5) of Proposition 2.1, $C_\infty := \cap_{n\geq 1} C_n$ is non-empty, compact and convex and clearly $T(C_\infty) \subset C_\infty$. By making appeal to Schauder fixed point theorem (see, for instance, [3, Theorem I.2.1]), T has some fixed point in $C_\infty \subset C$.

Next, as S is affine, we have

$$S(\overline{\text{Conv}}(B)) \subset \overline{S(\text{Conv}(B))} = \overline{\text{Conv}}(S(B)), \quad \text{for all non-empty } B \subset C. \tag{3.6}$$

Let C_n and C_∞ as above. We prove, by induction, that

$$S(C_n) \subset C_n, \quad \text{for all } n \geq 1. \tag{3.7}$$

Indeed, from (3.6) we obtain

$$S(C_1) = S(\overline{\text{Conv}}(T(C))) \subset \overline{\text{Conv}}(S(T(C))) = \overline{\text{Conv}}(T(S(C))) \subset \overline{\text{Conv}}(T(C)) = C_1,$$

and therefore (3.7) holds for $n = 1$. Assuming (3.7) remains true for $n - 1$, for a given $n > 1$, and bearing in mind (3.6) we have

$$S(C_n) = S(\overline{\text{Conv}}(T(C_{n-1}))) \subset \overline{\text{Conv}}(S(T(C_{n-1}))) = \overline{\text{Conv}}(T(S(C_{n-1}))) \subset$$

$$\overline{\text{Conv}}(T(C_{n-1})) = C_n,$$

and consequently, (3.7) holds as claimed.

Note that

$$S(C_\infty) = S\big(\bigcap_{n\geq 1} C_n\big) \subset \bigcap_{n\geq 1} S(C_n) \subset \bigcap_{n\geq 1} C_n = C_\infty,$$

and therefore, again by the Schauder fixed point theorem, $F_S := \{x \in C : S(x) = x\} \neq \emptyset$.

As $T(x^*) = T(S(x^*)) = S(T(x^*))$ for all $x^* \in F_S$, we find that $T(F_S) \subset F_S$. Moreover, F_S is closed (by the continuity of S) and convex because of S is affine. Thus, replacing C by F_S and reasoning as above, T has some fixed point in F_S which, of course, is also a fixed point of S and the proof is now complete.

□

Some comments are necessary before to continue.

(I) If S is the identity mapping, inequality (3.5) is redundant as it follows directly from Proposition 2.1, and the above result becomes, precisely, into Theorem 2.3.

(II) The class of affine mappings obeying (3.5) is large. Indeed, given $x_0 \in X$ and $c \in \mathbb{R}$ such that $x_0 + cC \subset C$, in view of Proposition 2.1 the affine mapping $S(x) := x_0 + cx$ for all $x \in C$ satisfies (3.5).

(III) From Example 2.4, we derive that the fixed point results based in the MNCs and those based in the DND ϕ are, essentially, different. There are results, in forms similar to Theorem 3.3 but based in the MNCs, to guarantee the existence of common fixed points of two commuting mappings; see, for instance, [**14, 16, 19**] and references therein. However, we will show below (see Example 3.4) that in general such results are essentially different to those based in the DND ϕ, and more specifically different to Corollary 3.4 proved below.

We show an example to illustrate Theorem 3.3.

Example 3.3. Let $C(I)$ be the Banach space of the continuous functions $x : I \longrightarrow \mathbb{R}$, endowed its usual supremum norm, and $U_{C(I)}$ its closed unit ball. Fixed two numbers β_T and β_S with $0 < 2\beta_T < \beta_S < 1$, consider the mappings $T, S : U_{C(I)} \longrightarrow U_{C(I)}$ defined by

$$T(x)(t) := \beta_T x(t) + (1-\beta_T)\int_0^t |x(s)|ds \quad \text{and } S(x)(t) := \beta_S x(t),$$

for all $x \in U_{C(I)}$ and $t \in I$. It is immediate to check that T and S are continuous and commute. By putting $\mathbf{1}(t) := 1$ and $\mathbf{0}(t) := 0$ for all $t \in I$, since

$$\|T(\mathbf{1}) - T(\mathbf{0})\| = 1 + \beta_T > 1 > \beta_S = \|S(\mathbf{1}) - S(\mathbf{0})\|,$$

we find that T is not a (S, λ)-contraction, for any $\lambda \in (0, 1)$.

A direct application of the Arcelá-Ascoli theorem, shows that the mapping $x \mapsto \int_0^t |x(s)|ds$ for all $x \in U_{C(I)}$ is precompact, that is, maps bounded sets into precompact sets (see also [3, Example I.3]). So, noticing Proposition 2.2 and the properties of the Hausdorff MNC χ, we have

$$\phi(T(B)) \leq 2\chi(T(B)) \leq 2\beta_T\chi(B) \leq 2\beta_T\phi(B), \quad \text{for all } B \subset U_{C(I)} \cap \mathcal{B}_{\text{arc}}(X). \tag{3.8}$$

Also, by virtue of property (B-1) of Proposition 2.1, the equality

$$\phi(S(B)) = \beta_S\phi(B), \quad \text{for all } B \subset U_{C(I)} \cap \mathcal{B}_{\text{arc}}(X). \tag{3.9}$$

holds. Then, joining (3.8) and (3.9), we have

$$\phi(T(B)) \leq \frac{2\beta_T}{\beta_S}\beta_S\phi(B) = \frac{2\beta_T}{\beta_S}\phi(S(B)), \quad \text{for all } B \subset U_{C(I)} \cap \mathcal{B}_{\text{arc}}(X),$$

and therefore T is a $(S, 2\beta_T/\beta_S)$-DND contractive mapping. Thus, as S is affine and, in view of the point (II) of the above comments, satisfies (3.5), from Theorem 3.3 we conclude that T and S have some common fixed point.

On the other hand, from Theorem 3.3 we can derive another generalization of Theorem 2.3:

Corollary 3.4. *Let $C \in \mathcal{B}(X)$ closed and convex and $T, S : C \longrightarrow C$ two continuous mappings such that T and S commute, S is an affine mapping satisfying (3.5) and TS is a DND-(S,λ)-contractive mapping, for some $\lambda \in (0,1)$. Then, T and S have some common fixed point.*

Proof. Define $R := TS : C \longrightarrow C$. Then is clear that R is continuous and commute with S. Also, by Theorem 3.3, R and S have some common fixed point. Then, if x^* is a such common fixed point, we have

$$x^* = S(x^*) = R(x^*) = T(S(x^*)) = T(x^*),$$

and the result follows. □

The above result was proved in [14, Theorem 2.1] and [16, Corollary 2] replacing the DND ϕ for a MNC μ and the contractiveness condition (iii) of Definition 3.1 by this one

$$\mu(T(S(B))) \leq \lambda\mu(S(B)), \quad \text{for all non-empty } B \subset C, \tag{3.10}$$

for some $\lambda \in (0,1)$, and omitting condition (3.5).

As we have pointed out above and in Section 2 (see Example 2.4), the Darbo fixed point theorem and Theorem 2.3 are different, because of the DND ϕ is not a MNC. So, one can expect that Corollary 3.4 and the above mentioned result be, essentially, different. This assert is evidenced in the following example.

Example 3.4. Let $C(I)$ as in Example 3.3, C and $T_0 : C \longrightarrow C$ the affine mapping and the bounded, convex, closed constructed in [28] which satisfies the following properties:

(a) $\|T_0(x) - T_0(y)\| = \|x - y\|$ for all $x, y \in C$ and $T_0(\mathbf{0}) = \mathbf{0} \in C$, $\mathbf{0}$ being the identically null function.

(b) There are two sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subset C$ with $\|x_n\| = \|y_n\| = 1$ and $T_0(x_n) = y_n$, for all $n \geq 1$.
(c) $\chi(T_0(B_0)) = 1 > 1/2 = \chi(B_0)$, for $B_0 := \{x_n : n \geq 1\} \subset C$.

Define $T, S : C \longrightarrow C(I)$ as $T(x)(t) := \frac{1}{2}T_0(x)(t)$ and $S(x)(t) := \frac{1}{2}x(t)$, for all $x \in C$ and $t \in I$. Noticing (a) and the above considerations, is clear that:

(i) $T(C) \subset C$, because of $T_0(\mathbf{0}) = \mathbf{0} \in C$, and $S(C) \subset C$.
(ii) T and S are continuous and commute. Also, S is affine and satisfies (3.5).
(iii) $\|T(x) - T(y)\| = \frac{1}{2}\|x - y\|$ for all $x, y \in C$.

From (b), for all $n \geq 1$ we have

$$\|T(x_n) - T(\mathbf{0})\| = \frac{1}{2}\|T_0(x_n)\| = \frac{1}{2}\|y_n\| = \frac{1}{2} = \frac{1}{2}\|x_n\| = \|S(x_n) - S(\mathbf{0})\|,$$

and so, T is not a (S, λ)-contractive mapping for any $\lambda \in (0, 1)$.

Let any $B \subset C \cap \mathcal{B}_{\mathrm{arc}}(X)$. From (iii), if γ is an α-dense curve in B, for some $\alpha > \phi(B)$, is clear that $T \circ \gamma$ is an $\alpha/2$-dense curve in $T(B)$. Therefore, $\phi(T(B)) \leq \phi(B)/2$, and consequently $\phi(T(S(B))) \leq \phi(S(B))/2$. Thus, conditions of Corollary 3.4 are fulfilled.

But, noticing (c) we infer that

$$\chi(T(S(B_0))) = \chi(T(\frac{1}{2}B_0)) = \frac{1}{4}\chi(T_0(B_0)) = \frac{1}{4} = \chi(S(B_0)),$$

where χ is the Hausdorff MNC. Then, the contractiveness condition (3.10) is not satisfied for the Hausdorff MNC χ.

4. Final remarks

In this paper we have proved, under suitable conditions and using the DND ϕ, some common fixed point results. By the exposed examples, we have shown that our main result and Theorem 1.2 are, essentially, different. Also, as direct consequence of our main result, we have derived a result, Corollary 3.4, which works in some cases where the *analogous* result based in the MNCs not. However, we can do some considerations which can be into account to improve, perhaps in future works, the exposed results.

On the one hand, condition (i) of Definition 1.1, namely, that T and S commute, can be replaced by a weaker one to prove the existence of common fixed points of two mappings (see, for instance, [**2**] and references therein). So, a possible generalization of our results could consist in replacing (i) of Definition 3.1 by a weaker one.

On the other hand, one of the most important generalizations of Darbo fixed point, due to Sadovskiĭ, is that based in the so called *condensing mappings* (see [**1**, **3**]). Then, another way to generalize Theorem 3.3 could be replace condition (iii) of Definition 3.1 by this one

$$\phi(T(B) < \phi(S(B)), \quad \text{for all } B \subset C \cap \mathcal{B}_{\mathrm{arc}}(X) \text{ with } \phi(B) > 0,$$

or even by another contractiveness condition like the used in [**14**, **16**, **19**] for the MNCs.

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References

[1] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, Operator Theory: Advances and Applications, vol. 55, Birkhauser Verlag, Basel, 1992.

[2] M. A. Alghamdi, S. Radenović and N. Shahzad, *On some generalizations of commuting mappings*, Abstr. Appl. Anal., **2012** (2012) Article ID 952052.

[3] J. M. Ayerbe Toledano, T. Domínguez Benavides, and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Operator Theory: Advances and Applications, vol. 99, Birkhauser Verlag, Basel, 1997.

[4] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales*, Fundam. Math., **3** (1922), 133-181.

[5] N. H. Bouzara and V. Karakaya, *Some common fixed point results for commuting k-set contraction mappings and their application*, Adv. Fixed Point Theory **6** (2016), 224–240.

[6] W. M. Boyce, *Commuting functions with no common fixed point*, Trans. Amer. Math. Soc., **137** (1969), 77-92.

[7] Y. Cherruault and G. Mora, *Optimisation Globale, Théorie des Courbes α-denses*, Económica, Paris 2005.

[8] C. Derbazi, Z. Baitiche, M. Benchohra, and Y Zhou, *Boundary value problem for ψ-Caputo fractional differential equations in Banach spaces via densifiability techniques*, Mathematics **10**(1) (2022), 153.

[9] G. García, *Existence of solutions for infinite systems of ordinary differential equations by densifiability techniques*, Filomat **32** (2018), 3419-3428.

[10] G. García, *Solvability of initial value problems with fractional order differential equations in Banach spaces by α-dense curves*, Fract. Calc. Appl. Anal., **20** (2017), 646-661.

[11] G. García and G. Mora, *Projective limits of generalized scales of Banach spaces and applications*, Ann. Funct. Anal., **13** (2022), 76.

[12] G. García and G. Mora, *A fixed point result in Banach algebras based on the degree of nondensifiability and applications to quadratic integral equations*, J. Math. Anal. Appl., **472** (2019), 1220–1235.

[13] G. García and G. Mora, *The degree of convex nondensifiability in Banach spaces*, J. Convex Anal., **22** (2015), 871–888.

[14] A. Hajji, *A generalization of Darbo's fixed point and common solutions of equations in Banach spaces*, Fixed Point Theory Appl., **63** (2013), 2013.

[15] J. P. Huneke, *On common fixed points of commuting continuous functions on an interval*, Trans. Amer. Math. Soc., **139** (1969), 371–381.

[16] M. El Harrak and A. Hajji, *A new common fixed point theorem for three commuting mappings*, Axioms **9** (2020), 105.

[17] H. Işik, B. Mohammadi, M. R. Haddadi and V. Parvaneh, *On a new generalization of Banach contraction principle with application*, Mathematics **7** (2019), 8 pages.

[18] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Montly **83** (1976), 261-263.

[19] N. Khodabakhshi and S. M. Vaezpour, Common fixed point theorems via measure of noncompactness, Fixed Point Theory **17** (2016), 381-386.

[20] B. Mohammadi, V. Parvaneh, and H. Hosseinzadeh, *Some Wardowski-Mizogochi-Takahashi-Type generalizations of the multi-valued version of Darbo's fixed point theorem with applications*, Int. J. Nonlinear Anal. Appl. doi: 10.22075/ijnaa.2023.23057.2470

[21] B. Mohammadi, A.A. Shole Haghighi, M. Khorshidi. M. De la Sen, and V. Parvaneh, *Existence of solutions for a system of integral equations using a generalization of Darbo's fixed point theorem*, Mathematics **8** (2020), 492.

[22] G. Mora, *The Peano curves as limit of α-dense curves*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **99** (2005), 23-28.

[23] G. Mora and Y. Cherruault, *Characterization and generation of α-dense curves*, Comput. Math. Appl., **33** (1997), 83-91.

[24] G. Mora and J.A. Mira, *Alpha-dense curves in infinite dimensional spaces*, Int. J. Pure and Appl. Math. **5** (2003), 257-266.

[25] G. Mora and D.A. Redtwitz, *Densifiable metric spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **105** (2011), 71-83.

[26] H. Sagan, *Space-filling Curves*, Springer-Verlag, New York, 1994.

[27] J. Vujaković, L. Ljajko, S. Radojević, and S. Radenović, *On some new Jungck–Fisher–Wardowski type fixed point results*, Symmetry **12** (2020).

[28] A. Wiśnicki, *An example of a nonexpansive mapping which is not 1-ball-contractive*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **LIX** (2005), 141-146.

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