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2-simultaneous quasi-Chebyshev and weakly-Chebyshev subspaces in quotient generalized 2-normed spaces

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Abstract. In this paper, we present characterizations of 2-simultaneous quasi-Chebyshevity and 2-simultaneous weakly Chebyshevity in quotient generalized 2-normed spaces.

1. Introduction and preliminaries

One of generalization of normed spaces is 2-normed spaces that play a critical role in functional analysis. The concept of linear 2-normed spaces was firstly introduced by Gähler [9] in 1965, and since then, many others have studied and written about it [7, 8]. Z. Lewandowska introdused a generalization of Gähler 2-normed spaces under the name of generalized 2-normed space and she investigated some of their characteristics between 1999 and 2006 [12]-[17].

Many mathematicians are interested by the idea of the best approximation and its different versions considering its importance in functional analysis [10, 18, 19]. Some authors (such as [2, 1, 4, 20, 21, 22]) have recently produced results on best approximation theory in generalized 2-normed spaces. The theory of best simultaneous approximation is one kind of best approximation for which significant findings have been obtained (for instance [5, 6, 11, 19]). Best simultaneous approximation in quotient normed spaces is studied by M. Iranmanesh and H. Mohebi in [11], and a characterization of 2-simultaneous pseudo-Chebyshevity in quotient generalized 2-normed spaces is studied by M. Abrishami-Moghaddam and T. Sistani in [3]. In

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this article, we will use elements from the dual space to introduce and analyze 2-simultaneous quasi-Chebyshev and 2-simultaneous weakly-Chebyshev subspaces of generalized 2-normed spaces with respect to a 2-bounded set, and then examine their transmission to and from quotient spaces.

Definition 1.1. [12, 13] Let X and Y be linear spaces, D be a non-empty subset of $X \times Y$ such that for every $x \in X$ and $y \in Y$, the sets

$$D_x = \{y \in Y : (x, y) \in D\} \ ; \ D_y = \{x \in X : (x, y) \in D\}$$

are linear subspaces of the spaces Y and X, respectively. A function $\|.,.\| : D \to [0, \infty)$ is called a *generalized 2-norm* on D if it satisfies the following conditions:
$(N1)\|\alpha x, y\| = |\alpha|\|x, y\| = \|x, \alpha y\|$ for all $(x, y) \in D$ and every scalar α.
$(N2)\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $(x, y), (x, z) \in D$.
$(N3)\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $(x, z), (y, z) \in D$.
Then $(D, \|.,.\|)$ is called a 2-normed set. In particular, if $D = X \times Y$, $(X \times Y, \|.,.\|)$ is called a *generalized 2-normed space*. Moreover, if $X = Y$, then generalized 2-normed space is denoted by $(X, \|.,.\|)$.

Definition 1.2. [16] Let X be a real linear space. Denote by $\mathcal{X}$ a non empty subset of $X \times X$ with the property $\mathcal{X} = \mathcal{X}^{-1}$(Symmetric) and such that the set $\mathcal{X}^y = \{x \in \mathcal{X}; (x; y) \in \mathcal{X}\}$ is a linear subspace of X, for all $y \in X$. A function $\|.,.\| : \mathcal{X} \to [0, \infty)$ satisfying the following conditions:
$(S1)\|x, y\| = \|y, x\|$ for all $(x; y) \in \mathcal{X}$,
$(S2)\|\alpha x, y\| = |\alpha|\|x, y\| = \|x, \alpha y\|$ for any real number α and all $(x, y) \in \mathcal{X}$,
$(S3)\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$ such that $(x, y), (x, z) \in \mathcal{X}$,
will be called a *generalized symmetric 2-norm* on $\mathcal{X}$. The set $\mathcal{X}$ is called a *symmetric 2-normed set*. In particular, if $\mathcal{X} = X \times X$, the function $\|.,.\|$ will be called a generalized symmetric 2-norm on X and the pair $(X; \|.,.\|)$ a generalized symmetric 2-normed space.

The following examples are some generalized 2-normed spaces and symmetric generalized 2-normed spaces.

Example 1.3. [17] 1) Let X be a real linear space having two norms $\|.\|_1$ and $\|.\|_2$. Then $(X, \|.,.\|)$ is a generalized 2-normed space with the 2-norm defined by

$$\|x, y\| = \|x\|_1.\|y\|_2 \ ; x, y \in X.$$

Specially if $\|.\|_1 = \|.\|_2$, our generalized 2-normed space will be a generalized symmetric 2-normed space.
2) Let X be a real inner product space. Then X is a symmetric generalized 2-normed space under the 2-norm

$$\|x, y\| = |\langle x, y\rangle| \ ; \ \forall \ x, y \in X.$$

3) Let X be the linear space of all sequence of real numbers. Put

$$\|x,y\| = \sum_{1}^{\infty} |x_n||y_n|,$$

where $x = \{x_n\}, y = \{y_n\} \in X$. Then $D = \{(x,y) \in X \times X \ : \ \|x,y\| < \infty\}$ is a symmetric 2-normed set and the function $\|.,.\| : D \to [0,\infty)$ is a generalized symmetric 2-normed on D.
4) Let A be a Banach algebra and $\|a,b\| = \|ab\|$ for all $a, b \in A$. Then, $(A, \|.,.\|)$ is a generalized 2-normed space.

Definition 1.4. Let $X \times Y$ be a generalized 2-normed linear space. $S_1 \times S_2$ is called a *2-bounded* subset of $X \times Y$ if there exists $r > 0$ such that $\|s_1, s_2\| < r$ for all $(s_1, s_2) \in S_1 \times S_2$.

Lemma 1.1. *Let $(X, \|.\|)$ be a normed space, and let X be equipped with the following generalized 2-norm*

$$\|x,y\| = \|x\|.\|y\|; \ \forall x, y \in X.$$

If S is a bounded set in X, then $S \times S$ is a bounded set in $X \times X$.

Proof. Let S be a bounded set in X. Then there exists $r > 0$ such that $\|x\| < r$, for each $x \in S$. Then we have

$$\|x,y\| = \|x\|.\|y\| < r.r = r^2,$$

for each $x, y \in S$. Therefore $S \times S$ is a 2-bounded set in $X \times X$. □

Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space and $W_1 \times W_2$ be a linear subspaces of $X \times Y$. A 2-functional $f : W_1 \times W_2 \to \mathbb{R}$ is called a *bilinear 2-functional* on $W_1 \times W_2$, whenever for all $x_1, x_2 \in W_1, y_1, y_2 \in W_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$;
i) $f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2)$,
ii) $f(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1 \lambda_2 f(x_1, y_1)$.
The set of all continuations bilinear 2-functional will be denoted by $(X \times Y)^*$ and it is called the dual space of $X \times Y$. Also we say that $f \in (M_1 \times M_2)^{\perp}$ when $f\mid_{M_1 \times M_2} = 0$.
The operator $\pi : X \times Y \longrightarrow \frac{X}{M_1} \times \frac{Y}{M_2}$ which is defined by $\pi(x,y) = (x + M_1, y + M_2)$, is called the *canonical map*.

Definition 1.5. [8] 1) The sequence $\{(x_n, y_n)\}$ in a generalized 2-normed space $X \times Y$ is called a *2-converges sequence* to (x, y) if

$$\lim_{n\to\infty} \|x_n - x, b\| = 0; \quad \forall\ b \in Y,$$

and

$$\lim_{n\to\infty} \|a, y_n - y\| = 0; \quad \forall\ a \in X,$$

and we write $(x_n, y_n) \longrightarrow (x, y)$.

2) The sequence $\{(x_n, y_n)\}$ in a generalized 2-normed space $X \times Y$ is called a *weakly 2-converges sequence* to (x, y) if $\lim f(x_n, y_n) = f(x, y)$ holds for each $f \in (X \times Y)^*$.

Definition 1.6. Let $X \times Y$ be a generalized 2-normed linear space. A subset $S_1 \times S_2 \subseteq X \times Y$ is called *compact (weakly compact)*, if every sequence $\{(x_n, y_n)\}$ in $S_1 \times S_2$ has a subsequence $\{(x_{n_k}, y_{n_k})\}$ which 2-converges (weakly 2-converges) to an element $(x_0, y_0) \in S_1 \times S_2$.

Let $X \times Y$ be a generalized 2-normed linear space, $W_1 \times W_2$ a subset of $X \times Y$ and $S_1 \times S_2$ a 2-bounded subset of $X \times Y$. We define

$$d(S_1 \times S_2, W_1 \times W_2) = \inf_{(w_1,w_2)\in W_1\times W_2} \sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_1, s_2 - w_2\|,$$

if there exists some $(w_1, w_2) \in W_1 \times W_2$ such that $\sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_1, s_2 - w_2\| < \infty$. $S_1 \times S_2$ is called *2-simultaneous proximinal* if for every $(s_1, s_2) \in S_1 \times S_2$ there exists an element $(w_{01}, w_{02}) \in W_1 \times W_2$ such that

$$d(S_1 \times S_2, W_1 \times W_2) = \sup_{(s_1,s_2)\in S_1\times S_2} \|s_1 - w_{01}, s_2 - w_{02}\|.$$

In this case $(w_{01}, w_{02}) \in W_1 \times W_2$ is called a *2-best simultaneous approximation* to $S_1 \times S_2$ from $W_1 \times W_2$. The set of all 2-best simultaneous approximation to $S_1 \times S_2$ from $W_1 \times W_2$ will be denoted by $\mathbf{S}_{W_1\times W_2}(S_1 \times S_2)$. If $S_1 \times S_2 = \{(x, y)\}$ where $(x, y) \in X \times Y$ then $\mathbf{S}_{W_1\times W_2}(S_1 \times S_2)$ is the set of all *2-best approximation* of (x, y) in $W_1 \times W_2$ that denoted by $P_{W_1\times W_2}(x, y)$ and also $W_1 \times W_2$ is called a *2-proximinal* subspace of $X \times Y$.

Let $M_1 \times M_2$ be a subspace of a generalized 2-normed linear space $X \times Y$ and $f \in (X \times M_2)^\perp \cap (M_1 \times Y)^\perp$. Define a bilinear 2-functional T_f on $\frac{X}{M_1} \times \frac{Y}{M_2}$ by $T_f(x + M_1, y + M_2) = f(x, y)$ for all $(x + M_1, y + M_2) \in \frac{X}{M_1} \times \frac{Y}{M_2}$. Then $T_f \in (\frac{X}{M_1} \times \frac{Y}{M_2})^*$ (the dual space of $\frac{X}{M_1} \times \frac{Y}{M_2}$).

Definition 1.7. Let $X \times Y$ be a generalized 2-normed linear space, $W_1 \times W_2$ a subspace of $X \times Y$ and $S_1 \times S_2$ a 2-bounded set in $X \times Y$.Then, $W_1 \times W_2$ is called

(i) *2-simultaneous quasi-Chebyshev* subspace if $\mathbf{S}_{W_1\times W_2}(S_1 \times S_2)$ is compact subset of $W_1 \times W_2$ for all 2-bounded subset $S_1 \times S_2$ in $X \times Y$.

(ii) *2-simultaneous weakly-Chebyshev* subspace if $\mathbf{S}_{W_1\times W_2}(S_1 \times S_2)$ is weakly compact subset of $W_1 \times W_2$ for all 2-bounded subset $S_1 \times S_2$ in $X \times Y$.

Theorem 1.2 ([2]). *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed linear space, and M_1 and M_2 be subspaces of X and Y respectively. Define*

$$\|.,.\| : \frac{X}{M_1} \times \frac{Y}{M_2} \longrightarrow [0, +\infty)$$

$$\|x + M_1, y + M_2\| = \inf_{(m_1,m_2)\in M_1\times M_2} \|x + m_1, y + m_2\|$$

for every $x \in X$ *and* $y \in Y$. *Then* $\|.,.\|$ *is a generalized 2-norm on* $\frac{X}{M_1} \times \frac{Y}{M_2}$.

In [2], the authors have shown that $\|.,.\|$ is a generalized 2-norm, which it is not necessary satisfying the conditions of 2-norm.

2. Main Results

We present characterizations of 2-simultaneous quasi-Chebyshevity and weakly-Chebyshevity in this section. Also, we shall use the following lemmas in the sequel which has been proved in [3].

Lemma 2.1. [3] *Let* $X \times Y$ *be a generalized 2-normed linear space and* $M_1 \times M_2$ *a 2-proximinal subset of* $X \times Y$.*Then for each nonempty 2-bounded subset* $S_1 \times S_2$ *in* $X \times Y$ *we have*

$$d(S_1 \times S_2, M_1 \times M_2) = \sup_{(s_1,s_2)\in S_1\times S_2} \inf_{(w_1,w_2)\in M_1\times M_2} \|s_1 - m_1, s_2 - m_2\|.$$

Lemma 2.2. [3] *Let* $W_1 \times W_2$ *be a 2-simultaneous proximinal subspace of generalized 2-normed space* $X \times Y$, $M_1 \times M_2$ *a 2-proximinal subspace of* $X \times Y$ *and* $M_1 \times M_2 \subseteq W_1 \times W_2$. *Then for each nonempty 2-bounded set* $S_1 \times S_2$ *with* $M_1 \times M_2 \subseteq S_1 \times S_2 \subseteq X \times Y$ *we have*

$$d(S_1 \times S_2, W_1 \times W_2) = d\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}, \frac{W_1}{M_1} \times \frac{W_2}{M_2}\Big).$$

Lemma 2.3. [3] *Let* $W_1 \times W_2$ *be a 2-simultaneous proximinal subspace of generalized 2-normed space* $X \times Y$, $M_1 \times M_2$ *a 2-proximinal subspace of* $X \times Y$, $S_1 \times S_2$ *a 2- bounded set in* $X \times Y$, $M_1 \times M_2 \subseteq W_1 \times W_2$. *Then,*

$$\pi\Big(\boldsymbol{S}_{W_1\times W_2}(S_1 \times S_2)\Big) \subseteq \boldsymbol{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big).$$

Lemma 2.4. [3] *Let* $W_1 \times W_2$ *be a 2-simultaneous proximinal subspace of generalized 2-normed space* $X \times Y$, $M_1 \times M_2$ *a 2-proximinal subspace of* $X \times Y$, $S_1 \times S_2$ *a 2-bounded set in* $X \times Y$, $M_1 \times M_2 \subseteq W_1 \times W_2$. *If* $(w_{01} + M_1, w_{02} + M_2) \in \boldsymbol{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big)$ *and* $(m_{01}, m_{02}) \in \boldsymbol{S}_{M_1\times M_2}(S_1 - w_{01}, S_2 - w_{02})$, *then* $(w_{01} + m_{01}, w_{02} + m_{02}) \in \boldsymbol{S}_{W_1\times W_2}(S_1 \times S_2)$.

Corollary 2.5. [3] *Let* $W_1 \times W_2$ *be a 2-simultaneous proximinal subspace of generalized 2-normed space* $X \times Y$, $M_1 \times M_2$ *a 2-proximinal subspace of* $X \times Y$, $S_1 \times S_2$ *a 2-bounded set in* $X \times Y$ *and* $M_1 \times M_2 \subseteq W_1 \times W_2$. *Then,*

$$\pi\Big(\boldsymbol{S}_{W_1\times W_2}(S_1 \times S_2)\Big) = \boldsymbol{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}\Big(\frac{S_1}{M_1} \times \frac{S_2}{M_2}\Big).$$

Now, we are ready to state and prove our main results.

Theorem 2.6. *Let $M_1 \times M_2$ and $W_1 \times W_2$ are subspaces of generalized 2-normed linear space $X \times Y$ such that $M_1 \times M_2$ is finite dimensional and 2-proximinal and $W_1 \times W_2$ is 2-simultaneous proximinal. Then the following are equivalent.*
(i) $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$ *is 2-simultaneous quasi-Chebyshev subspace of* $\frac{X}{M_1} \times \frac{Y}{M_2}$.
(ii) $(W_1 + M_1) \times (W_2 + M_2)$ *is 2-simultaneous quasi-Chebyshev subspace of* $X \times Y$.

Proof. $(i) \Rightarrow (ii)$ Let $S_1 \times S_2$ be a 2-bounded set in $X \times Y$ and $\{(a_n, b_n)\}$ a sequence in $\mathbf{S}_{\frac{W_1}{M_1} \times \frac{W_2}{M_2}}(S_1 \times S_2)$. Then by corollary 2.5, we have

$$(a_n + M_1, b_n + M_2) \in \mathbf{S}_{\frac{W_1+M_1}{M_1} \times \frac{W_2+M_2}{M_2}}(\frac{S_1}{M_1} \times \frac{S_2}{M_2}).$$

Since $\mathbf{S}_{\frac{W_1+M_1}{M_1} \times \frac{W_2+M_2}{M_2}}(\frac{S_1}{M_1} \times \frac{S_2}{M_2})$ is compact, there exists $(a_0, b_0) \in (W_1 + M_1) \times (W_2 + M_2)$ and a subsequence $\{(a_{n_k} + M_1, b_{n_k} + M_2)\}_{k \geq 1}$ of $\{(a_n + M_1, b_n + M_2)\})$ such that $(a_0 + M_1, b_0 + M_2) \in \mathbf{S}_{\frac{W_1+M_1}{M_1} \times \frac{W_2+M_2}{M_2}}(\frac{S_1}{M_1} \times \frac{S_2}{M_2})$ and

$$\{(a_{n_k} + M_1, b_{n_k} + M_2)\}_{k \geq 1} \longrightarrow (a_0 + M_1, b_0 + M_2).$$

Since $\|.,.\|$ is continuous, we have

$$\|a_{n_k} - a_0 + M_1, b_{n_k} - b_0 + M_2\| \longrightarrow 0. \tag{1}$$

Now, since $M_1 \times M_2$ is 2-proximinal in $X \times Y$, for each $k \geq 1$, there exists

$$(m_{1nk}, m_{2nk}) \in P_{M_1 \times M_2}(a_0 - a_{n_k}, b_0 - b_{n_k})$$

such that

$$\begin{aligned} \|a_0 - a_{n_k} - m_{1nk}, b_0 - b_{n_k} - m_{2nk}\| &= d((a_0 - a_{n_k}, b_0 - b_{n_k}), M_1 \times M_2) \\ &= \|a_0 - a_{n_k} + M_1, b_0 - b_{n_k} + M_2.\| \end{aligned} \tag{2}$$

Therefore from 1 and 2, we have

$$\lim_{k \to \infty} \|a_0 - a_{n_k} + m_{1nk}, b_0 - b_{n_k} + m_{2nk}\| = 0.$$

Since $\{\|a_0 - a_{n_k} + m_{1nk}, b_0 - b_{n_k} + m_{2nk}\|\}$ is a convergence sequence of real numbers, it is bounded and therefore the sequence $\{(a_0 - a_{n_k} + m_{1nk}, b_0 - b_{n_k} + m_{2nk})\}$ is a 2-bounded sequence. On the other hand, since $(a_n, b_n) \in \mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2)$ for all $n \geq 1$,$\{(a_{n_k}, b_{n_k})\}$ is 2-bounded sequence. Hence $\{(m_{1nk}, m_{2nk})\}$ is 2-bounded sequence in $M_1 \times M_2$. Since $M_1 \times M_2$ is finite dimensional subspace of $X \times Y$, without loss of generality we can assume that $\{(m_{1nk}, m_{2nk})\}$ 2-convergence to an element $(m_{01}, m_{02}) \in M_1 \times M_2$. Let $(a', b') = (a_0 - m_{01}, b_0 - m_{02})$. Thus, $(a', b') \in (W_1 + M_1) \times (W_2 + M_2)$. Now for each $b \in Y$ we have

$$\begin{aligned} \|a' - a_{n_k}, b\| &= \|a_0 - m_{01} - a_{n_k}, b\| \\ &\leq \|a_0 - a_{n_k} - m_{1_{n_k}}, b\| + \|m_{1_{n_k}} - m_{01}, b\| \end{aligned}$$

for all $k \geq 1$. Hence

$$\lim_{k\to\infty} \|a' - a_{n_k}, b\| = 0.$$

Similarly for each $a \in X$ we get

$$\lim_{k\to\infty} \|a, b' - b_{n_k}\| = 0.$$

Now by definition of 2-convergence sequences 1.5,

$$(a_{n_k}, b_{n_k}) \longrightarrow (a', b').$$

Since $(a_{n_k}, b_{n_k}) \in \mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1\times S_2)$ for all $n \geq 1$ and $\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1\times S_2)$ is closed, (a', b') is an element of $\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2)$.

Therefore, $\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2)$ is compact.

$(ii) \Rightarrow (i)$ Let $S_1 \times S_2$ be an arbitrary 2-bounded set in $X \times Y$ and $(W_1 + M_1) \times (W_2 + M_2)$ be a simultaneous 2-quasi-Chebyshev subspace of $X \times Y$. Then $\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2)$ is compact. But since the canonical map is continuous, so $\pi(\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2))$ is compact. Thus by corollary 2.5,

$$\begin{aligned}\pi(\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2)) &= \mathbf{S}_{\frac{W_1+M_1}{M_1}\times\frac{W_2+M_2}{M_2}}(\frac{S_1}{M_1} \times \frac{S_2}{M_2}) \\ &= \mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}(\frac{S_1}{M_1} \times \frac{S_2}{M_2}).\end{aligned}$$

Therefore, $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$ is simultaneous 2-quasi-Chebyshev subspace of $\frac{X}{M_1} \times \frac{Y}{M_2}$. $\square$

Corollary 2.7. *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space and let $M_1 \times M_2$ and $W_1 \times W_2$ are subspaces of $X \times Y$ such that $M_1 \times M_2 \subseteq W_1 \times W_2$. If $W_1 \times W_2$ is simultaneous 2-quasi-Chebyshev subspace of $X \times Y$, then $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$ is 2-simultaneous quasi-Chebyshev subspace of $\frac{X}{M_1} \times \frac{Y}{M_2}$.*

Corollary 2.8. *Let $M_1 \times M_2$ and $W_1 \times W_2$ are subspaces of generalized 2-normed linear space $X \times Y$ such that $M_1 \times M_2$ is finite dimensional and 2-proximinal, $W_1 \times W_2$ is simultaneous 2-proximinal and $M_1 \times M_2 \subseteq W_1 \times W_2$. Then the following are equivalent.*
(i) $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$ is 2-simultaneous quasi-Chebyshev subspace of $\frac{X}{M_1} \times \frac{Y}{M_2}$.
(ii) $W_1 \times W_2$ is 2-simultaneous quasi-Chebyshev subspace of $X \times Y$.

Theorem 2.9. *Let $M_1 \times M_2$ and $W_1 \times W_2$ are subspaces of generalized 2-normed linear space $X \times Y$ such that $M_1 \times M_2$ is finite dimensional and 2-proximinal and $W_1 \times W_2$ is simultaneous 2-proximinal. Then the following are equivalent.*
(i) $\frac{W_1}{M_1} \times \frac{W_2}{M_2}$ is simultaneous 2-weakly-Chebyshev subspace of $\frac{X}{M_1} \times \frac{Y}{M_2}$.
(ii) $(W_1 + M_1) \times (W_2 + M_2)$ is 2-simultaneous weakly-Chebyshev subspace of $X \times Y$.

PROOF. $(i) \Rightarrow (ii)$ Let $S_1 \times S_2$ be a 2-bounded set in $X \times Y$ and $\{(a_n, b_n)\}$ a sequence in $\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1 \times S_2)$. Then by lemma 2.3, $\{(a_n + M_1, b_n + M_2)\}$

is a sequence in

$$\mathbf{S}_{\frac{W_1+M_1}{M_1}\times\frac{W_2+M_2}{M_2}}(\frac{S_1}{M_1}\times\frac{S_2}{M_2})=\mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}(\frac{S_1}{M_1}\times\frac{S_2}{M_2}).$$

Since $\mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}(\frac{S_1}{M_1}\times\frac{S_2}{M_2})$ is weakly compact, there exists a subsequence $\{(a_{n_k}+M_1,b_{n_k}+M_2)\}$ of $\{(a_n+M_1,b_n+M_2)\}$ such that $\{(a_{n_k}+M_1,b_{n_k}+M_2)\}$ 2-convergence weakly to an element $(a_0+M_1,b_0+M_2)\in\mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}(\frac{S_1}{M_1}\times\frac{S_2}{M_2})$. But since $M_1\times M_2$ is a 2-proximinal, $(a_0+m_{01},b_0+m_{02})\in\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1\times S_2)$, for some $(m_{01},m_{02})\in M_1\times M_2$. But for every $f\in(X\times Y)^*$ we have $T_f\in(\frac{X}{M_1}\times\frac{Y}{M})^*$. Therefore,

$$\begin{aligned}f(a_{n_k},b_{n_k})&=T_f(a_{n_k}+M_1,b_{n_k}+M_2)\to T_f(a_0+m_{01}+M_1,b_0+m_{02}+M_2)\\&=f(a_0+m_{01},b_0+m_{02}).\end{aligned}$$

Hence, $\{(a_{n_k},b_{n_k})\}$ 2-convergence weakly to $(a_0+m_{01},b_0+m_{02})\in\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1\times S_2)$. Thus, $\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1\times S_2)$ is weakly compact and hence (W_1+M_1,W_2+M_2) is simultaneous 2-weakly-Chebyshev subspace of $X\times Y$.

$(ii)\Rightarrow(i)$ Let $S_1\times S_2$ be an arbitrary 2-bounded set in $X\times Y$ and $\{(w_{1n}+M_1,w_{2n}+M_2)\}$ a sequence in $\mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}(\frac{S_1}{M_1}\times\frac{S_2}{M_2})$. Since $M_1\times M_2$ is 2-proximinal and (S_1-w_{1n},S_2-w_{2n}) is a 2-bounded set in $X\times Y$ for all $n\geq1$, there exists $(m_{1n},m_{2n})\in\mathbf{S}_{M_1\times M_2}(S_1-w_{1n},S_2-w_{2n})$ for all $n\geq1$. But by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}&\sup_{(s_1,s_2)\in S_1\times S_2}\|s_1-w_{1n}-m_{1n},s_2-w_{2n}-m_{2n}\|\\&=\inf_{(m_1,m_2)\in M_1\times M_2}\sup_{(s_1,s_2)\in S_1\times S_2}\|s_1-w_{1n}-m_1,s_2-w_{2n}-m_2\|\\&=\sup_{(s_1,s_2)\in S_1\times S_2}\inf_{(m_1,m_2)\in M_1\times M_2}\|s_1-w_{1n}-m_1,s_2-w_{2n}-m_2\|\\&=\sup_{(s_1,s_2)\in S_1\times S_2}\|s_1-w_{1n}+M_1,s_2-w_{2n}+M_2\|\\&\leq d\Big(\frac{S_1}{M_1}\times\frac{S_2}{M_2},\frac{W_1}{M_1}\times\frac{W_2}{M_2}\Big)\\&\leq d(S_1\times S_2,(W_1+M_1)\times(W_2+M_2)).\end{aligned}$$

Therefore $\{(w_{1n}+m_{1n},w_{2n}+m_{2n})\}$ is a sequence in $\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1\times S_2)$. Since $\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1\times S_2)$ is weakly compact, there exists a subsequence $\{(w_{1n_k}+m_{1n_k},w_{2n_k}+m_{2n_k})\}$ of $\{(w_{1n}+m_{1n},w_{2n}+m_{2n})\}$ such that $\{(w_{1n_k}+m_{1n_k},w_{2n_k}+m_{2n_k})\}$ 2-convergence weakly to an element

$$(a_0,b_0)\in\mathbf{S}_{(W_1+M_1)\times(W_2+M_2)}(S_1\times S_2).$$

By corollary 2.5, $(a_0 + M_1, b_0 + M_2)$ is an element of $\mathbf{S}_{\frac{W_1+M_1}{M_1}\times\frac{W_2+M_2}{M_2}}(\frac{S_1}{M_1}\times\frac{S_2}{M_2}) = \mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}(\frac{S_1}{M_1}\times\frac{S_2}{M_2})$. Note that for every $f \in (\frac{X}{M_1}\times\frac{Y}{M})^*$ we have

$$\begin{aligned} &f(w_{1n_k} + M_1, w_{2n_k} + M_2) = f(w_{1n_k} + m_{1n_k} + M_1, w_{2n_k} + m_{2n_k} + M_2) \\ &\quad = f \circ \pi(w_{1n_k} + m_{1n_k}, w_{2n_k} + m_{2n_k}) \to f \circ \pi(a_0, b_0) = f(a_0 + M_1, b_0 + M_2). \end{aligned}$$

It follows that $\{(w_{1n_k}+M_1, w_{2n_k}+M_2)\}$ 2-convergence weakly to $(w_{01}+M_1, w_{02}+M_2)$. Hence, $\mathbf{S}_{\frac{W_1}{M_1}\times\frac{W_2}{M_2}}(\frac{S_1}{M_1}\times\frac{S_2}{M_2})$ is weakly compact subset of $\frac{X}{M_1}\times\frac{Y}{M_2}$. Therefore, $\frac{W_1}{M_1}\times\frac{W_2}{M_2}$ is 2-simultaneous weakly-Chebyshev subspace of $\frac{X}{M_1}\times\frac{Y}{M_2}$. □

Corollary 2.10. *Let $M_1 \times M_2$ and $W_1 \times W_2$ are subspaces of generalized 2-normed linear space $X\times Y$ such that $M_1\times M_2$ is finite dimensional and 2-proximinal, $W_1\times W_2$ is 2-simultaneous proximinal and $M_1\times M_2 \subseteq W_1\times W_2$. Then the following are equivalent.*
(i) $\frac{W_1}{M_1}\times\frac{W_2}{M_2}$ is 2-simultaneous weakly-Chebyshev subspace of $\frac{X}{M_1}\times\frac{Y}{M_2}$.
(ii) $W_1 \times W_2$ is 2-simultaneous weakly-Chebyshev subspace of $X \times Y$.

Conclusions

In this study, we develop 2-simultaneous quasi-Chebyshev and 2-simultaneous weakly-Chebyshev subspaces in generalized 2-normed spaces and investigate the conditions under which these subspaces are transmitted to and from quotient spaces. Future study on the concepts of 2-quasi cochebyshev, 2-weakly cochebyshev, (2,ε)-quasi chebyshev, (2,ε)-weakly chebyshev, 2-strong quasi chebyshev and 2-strong weaklychebyshev in quotient generalized 2-normed spaces is suggested.

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