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# On some new observations on Kurepa's left factorial

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Abstract. After a brief historical excursus, we prove in a simple way some properties of Kurepa's function, the left factorial. We introduce Kurepa's hypothesis, propose a new description, and the relation to Bezout's parameters and Diophantine equation. A numerical analysis supports Kurepa's hypothesis and the conjecture about distribution for Kurepa's function.

## 1. Introduction and preliminaries

In 1971, following his seminal works [11], [12], [13], Kurepa introduced [14] the left factorial function, with the symbol $!n$, where $n \in \mathbb{N}$, also known as Kurepa's function,

$$K(n) = !n = \sum_{i=0}^{n-1} i! = \sum_{i=0}^{n-1} \Gamma(i+1) \ , \tag{1}$$

and later [15] extended its definition to arguments $z$ on the complex plane $\Re(z) > 0$

$$K(z) = \int_0^{+\infty} e^{-t} \frac{t^z - 1}{t-1} dt \ ,$$

$$\Gamma(x+1) = \int_0^{+\infty} e^{-t} t^x dt \ . \tag{2}$$

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For $n \in \mathbb{N}$, $\Gamma(n+1) = n!$. The recurrence relation holds true:

$$\Gamma(x+1) = x\Gamma(x) \ , \tag{3}$$

and its asymptotic behavior (known as Stirling's formula) for $x \to +\infty$ is given by

$$\Gamma(x+1) \sim \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left[1 + \frac{1}{12x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right] \ . \tag{4}$$

Substituting (2) in (1) we obtain Kurepa's function on the complex plane:

$$K(n) = \int_0^{+\infty} e^{-t} \sum_{i=0}^{n-1} t^i dt = \int_0^{+\infty} e^{-t} \frac{t^n - 1}{t-1} dt$$

that is

$$K(x) = \int_0^{+\infty} e^{-t} \frac{t^x - 1}{t-1} dt \ . \tag{5}$$

The following theorems for the function $K(x)$ and its relation with function $\Gamma(x)$ were first established in [15]. Here we will show novel simple proofs for them.

**Theorem 1.1.** *The following relation holds true for $z \in \mathbb{C}$:*

$$K(z) = K(z+1) - \Gamma(z+1) \tag{6}$$

Proof. Using the expressions of (2) and (5) in (6) we obtain

$$\int_0^{+\infty} e^{-t} \frac{t^z - 1}{t-1} dt = \int_0^{+\infty} e^{-t} \left[\frac{t^{z+1} - 1}{t-1} - t^z\right] dt = \int_0^{+\infty} e^{-t} \left[\frac{t^z - 1}{t-1}\right] dt \ . \tag{7}$$

□

**Theorem 1.2.** *Kurepa's function has the following limit for $x \to +\infty$:*

$$\lim_{x \to +\infty} \frac{K(x)}{\Gamma(x)} = 1 \ . \tag{8}$$

Proof. From the definition (1) of $K(x)$, for $n \in \mathbb{N}$ we have:

$$K(n) = \sum_{i=0}^{n-1} i! \sim (n-1)! + (n-2)! + \cdots + 1! + 0! \ . \tag{9}$$

Inserting (9) in (8) we obtain

$$\lim_{n \to +\infty} \frac{K(n)}{\Gamma(n)} = \lim_{n \to +\infty} \frac{(n-1)! + (n-2)! + \dots}{(n-1)!} = \lim_{n \to +\infty} \left[1 + \mathcal{O}\left(\frac{1}{n-1}\right)\right] = 1 \ ,$$

this result obtained so far is valid for the case $n \in \mathbb{N}$ only. The integrand of (7) is well-defined for any $t \in \mathbb{R}^+$, so $K(z), z \in \mathbb{C}, \Re(z) > 0$ could be rewritten as

$$K(z) = \int_0^{+\infty} e^{-t} t^{z-1} \left[\sum_{n=0}^{+\infty} \left(\frac{1}{t^n} - \frac{1}{t^{n+z}}\right)\right] dt \sim \Gamma(z) + \Gamma(z-1) + \Gamma(z-2) + \dots \ , \tag{10}$$

the asymptotic behavior is given for $z \to \infty$, and using (3) one has

$$\lim_{z\to\infty} \frac{K(z)}{\Gamma(z)} = \lim_{z\to\infty} \frac{\Gamma(z) + \frac{\Gamma(z)}{z-1} + \mathcal{O}\left[\frac{\Gamma(z)}{(z-1)(z-2)}\right]}{\Gamma(z)} = 1 \; .$$

□

**Theorem 1.3.** *Kurepa's function has the following limit for $x \to +\infty$:*

$$\lim_{x\to+\infty} \frac{K(x)}{\Gamma(x+1)} = 0 \; .$$

Proof. Divide (6) by $\Gamma(x+1)$ obtaining

$$\lim_{x\to+\infty} \frac{K(x)}{\Gamma(x+1)} = \lim_{x\to+\infty} \frac{K(x+1)}{\Gamma(x+1)} - 1 = 1 - 1 = 0$$

by virtue of (8). □

## 2. Main results

In this section we present some equivalent statements for Kurepa's hypothesis, proposing some other new forms, and a suggestion for a new research direction by means of a linear Diophantine equation.

In the following section we present new numerical results supporting the conjecture of uniform distribution for $K(p)$.

**Hypothesis 1.** Kurepa's hypothesis is that the following, equivalent statements, hold true:

$$\begin{aligned} \gcd(K(n), n!) &= 2, & n &\geq 2, \\ \bmod\,(K(n), n) &\not\equiv 0 & n &> 2, \\ \bmod\,(K(p), p) &\not\equiv 0, & p &\geq 3, p \text{ prime number.} \end{aligned} \tag{11}$$

Let be $n \in \mathbb{N}, n \geq 2$, then, using the Euclidean algorithm for finding the $\gcd(!n, n!)$, the first formula of (11) could be rewritten as

$$x \cdot !n + y \cdot n! = 2 \; , \tag{12}$$

if such $x, y \in \mathbb{Z}$ exist. This is a Diophantine equation, and $(x, y)$ are the Bezout's parameters. From Kurepa's hypothesis, $\gcd(!n, n!) = 2$, therefore, equation (12) has an infinite number of solutions, that is Bezout parameters. If $(x_0, y_0)$ is a pair that solves (12), then follows that

$$x = x_0 + \frac{n!}{2}k \; , y = y_0 - \frac{!n}{2}k$$

with $k \in \mathbb{Z}$ are all Bezout's parameters that solve (12).

For instance, for $n = 4$ we have

$$x \cdot 10 + y \cdot 24 = 2$$

and $(x_0, y_0) = (5, -2)$. Then $x = 5 - 12k$ and $y = -2 + 5k$ are the Bezout's parameters of this equation.

Dividing (12) by 2 one could also observe that $!n/2$ and $n!/2$ are coprime numbers (in particular $!n/2$ is odd while $n!/2$ is even). Therefore, $!n/2$ has no factors smaller than $n$, thus recovering the second formula of (11).

First formula of (11) is also equivalent to:

$$\begin{aligned} \gcd(!!(2n), (2n)!!) =& 2, \\ \gcd(!!(2n+1), (2n+1)!!) =& 2 \end{aligned}$$

where we have defined $!!(2n) = \sum_{i=0}^{n-1} (2i)!!$ with $0!! = 0$, $!!(2n+1) = \sum_{i=0}^{n-1} (2i+1)!!$, and have used the relation $\gcd(a, b) \times \mathrm{lcm}(a, b) = a \times b$.

There are many equivalent forms for Kurepa's hypothesis (see [**9**] for more details), first one of (11) was stated in his original paper [**14**].

In 2004 the paper [**4**] presented a proof of Kurepa's hypothesis that was shown in 2011 to be wrong by the same authors [**5**], so they retracted the original paper. Until the present time (2022) there does not exist a real proof or a counterexample for (11).

## 3. Numerical analysis

We now present a numerical analysis and some new results of Kurepa's Hypothesis 1, starting from the last formula of (11). We have analyzed the first 11000 prime numbers $p$, from 3 to the $11000^{th}$ prime number being 116447, $!116447 \approx 116446! \approx 1.045 \times 10^{539361}$ (the first approximation is valid because of (8)). According to hypothesis found in [**1**], $K(p)$ modulo $p$ is a random number with random uniform distribution. In order to check this hypothesis we have shown the distribution of $\mod (K(p), p)$ and have compared it to random numbers, random$(p)$, with uniform distribution in $[0, p)$. In figure 1 the value of $\mod (K(p), p)$ is shown as a function of the prime $p$ for the first 11000 prime numbers. There is no value of $p$ for which $\mod (K(p), p) = 0$, thus Kurepa's hypothesis has been verified up to $p = 116447$. Figure 2 shows the normalized plot of $\mod (K(p), p)/p$, providing us with the more familiar rectangular uniform distribution of random numbers in $[0, p)$.

Figures 3 and 4 show respectively the same kind of plots as before, this time for a uniform distribution of random numbers random$(p)$ generated in the range $[0, p)$.

The comparison of the two group of results does not show substantial differences in distributions between the results of $\mod (K(p), p)$ and the random number generator random$(p)$. There is also no visible pattern that could possibly spoil the conjectured randomness of $\mod (K(p), p)$ in figures 1 and 2.

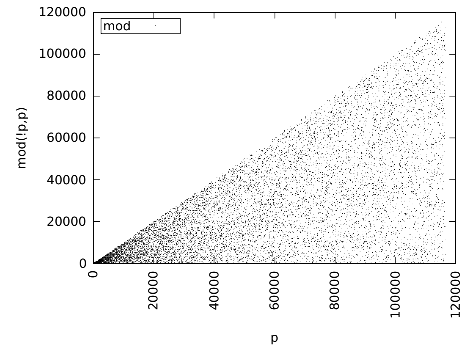


FIGURE 1. mod $(!p, p)$ versus $p$ for the first 11000 prime numbers

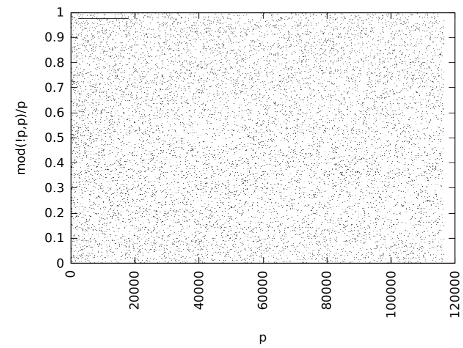


FIGURE 2. mod $(!p, p)/p$ versus $p$ for the first 11000 prime numbers

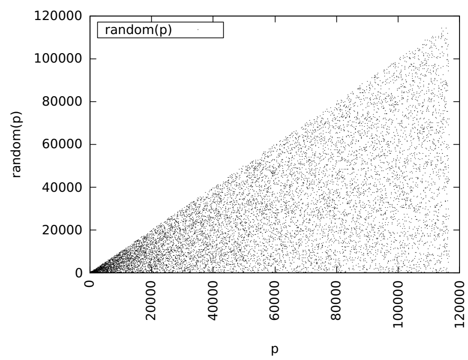


FIGURE 3. random$(p)$ versus $p$ for the first 11000 prime numbers

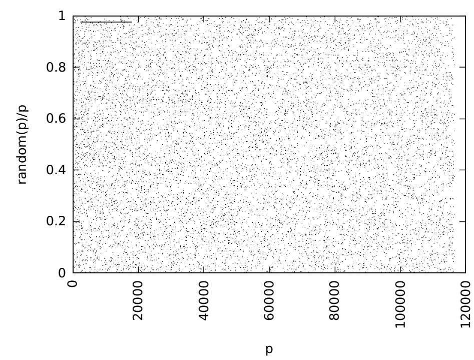


FIGURE 4. random$(p)/p$ versus $p$ for the first 11000 prime numbers

In figure 5 we compare the results of mod $(!p, p)/p$, shown in fig. 2, with respect to a uniform distribution between $[0, 1]$, as a function of number of primes taken into account. As expected, with increasing number of samples considered, that is with a larger number of primes, the distribution of mod $(!p, p)/p$ is getting closer to a uniform distribution, with an error smaller than 0.5%, which further decreases when the number of primes is larger than 7000.

This numerical analysis concludes that Kurepa's hypothesis

$$\text{mod } (K(p), p) \not\equiv 0$$

is true for all primes $p$ verified so far, and also that this value is randomly uniformly distributed in the range $[0, p)$.

In [1] the search for a counterexample of the conjecture was performed, without success, for $p < 2^{34} \approx 1.718 \times 10^{10}$ by means of GPU computing. There is also given a fairly exhaustive list of all numerical attempts to solve this problem.

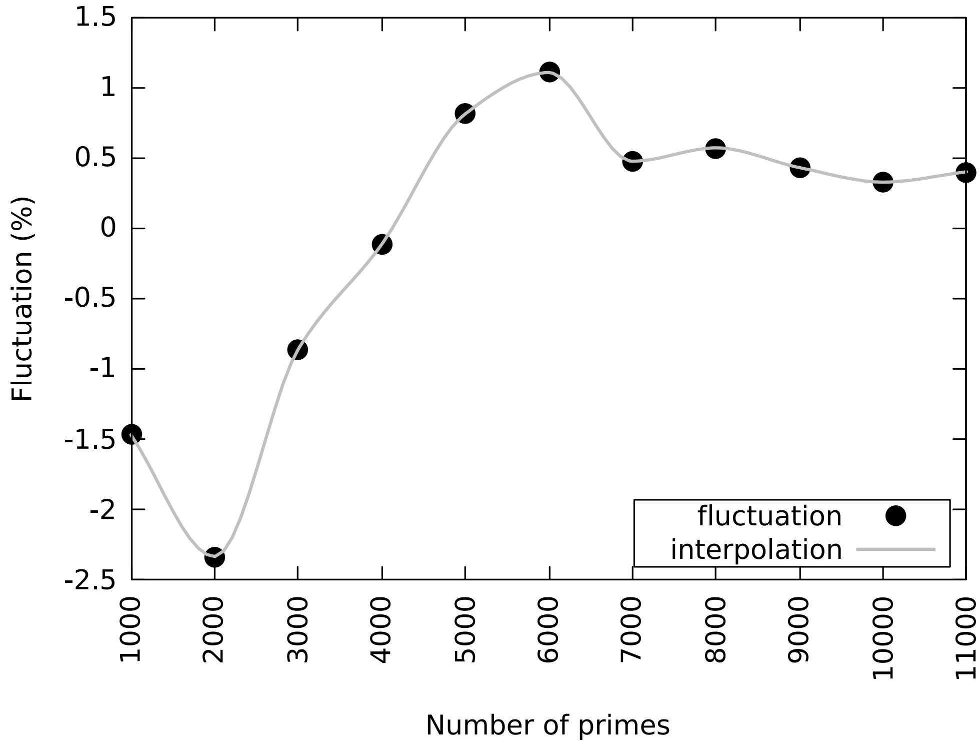


FIGURE 5. Difference of mod $(!p, p)/p$ from a uniform distribution, in percentage, with respect to number of primes

## 4. Conclusions

Fifty years have passed since the formulation of Kurepa's hypothesis [14]. Many attempts have been made in order to provide a solution to this problem, either by means of a rigorous theorem, with an approximate solution, or with a numerical approach [2], [3], [6], [8], [10], [16]-[38]. Various tentative solutions increase the importance of Kurepa's hypothesis, and considering its simple formulation it deserves to stay in the realm of other famous unsolved problems in number theory like Collatz conjecture [7], Goldback's conjecture, de Polignac's conjecture, Legendre's conjecture, Landau's problem, to name a few.

The present work has brought simplifications to some proofs concerning the Kurepa's function. There are some other equivalent statements of Kurepa's hypothesis, and a new study route using a linear Diophantine equation. Quantitative computer simulations support all the aforementioned hypotheses, and show also how the distribution of mod $(!p, p)/p$ approaches a uniform distribution - a fact not originally envisaged by Kurepa - with increasing prime $p$.

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