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Common fixed point results for $\omega-$compatible and $\omega-$weakly compatible maps in modular metric spaces

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Abstract. The aim of this paper is to prove a common fixed point theorem for two pairs of ω-compatible and ω-weakly compatible maps for extending and generalizing the results of Murthy and Prasad [13] in modular metric spaces. The main result is also illustrated by an example to demonstrate the degree of validity of our hypothesis.

1. Introduction

The metric fixed point theory is very important and useful in mathematics. It can be applied in various branches of mathematics, variational inequalities optimization and approximation theory. Polish mathematician Banach observed the first metric fixed point results in the setting of complete normed spaces. In 1976, Jungck [9] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. The self-maps f and g of a set Ω are called commutative if $fgu = gfu$ for all $u \in \Omega$. After that Sessa [17] introduced the notion of weakly commuting maps. "Let f and g be mappings from a (Ω, d) metric space into itself. The mappings f and g are said to be weakly commuting if $d(fgu, gfu) \leq d(fu, gu)$ for each u in Ω." Further, Jungck [10] introduced more generalized commutativity, the so-called compatibility, which is more general than that of weak commutativity. "Let (Ω, d) be a metric space and $f, g : \Omega \to \Omega$. The mappings f and g are said to be compatible if $\lim_{n\to+\infty} d\left(fgu_n, gfu_n\right) = 0$, whenever $\{u_n\}$ is a sequence in X such

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that $\lim_{n\to+\infty} f\mathrm{x_n} = \lim_{n\to+\infty} g\mathrm{x_n} = t$ for some $t \in \Omega$." After then in 1998, Jungck and Rhoades [11] introduced the notion of weakly compatible. "A pair of maps f and g is called weakly compatible pair if they commute at coincidence points that is, if $fu = gu$ implies $fgu = gfu''$ and showed that compatible maps are weakly compatible but converse need not be true. On the other hand, the notion of modular metric space was introduced by Chistyakov with the time parameter λ (say) and his purpose was to define the notion of a modular on an arbitrary set, develop the theory of metric spaces generated by modulars, called modular metric spaces in [4], [6],[7]. This is a generalization of the classical modular spaces like Orlicz spaces (see [12]). In recent years, there has been a great interest in the study of the fixed point property in modular metric spaces (see [1, 2, 16]). For more details on modular metric fixed point theory, the reader may consult the books [3, 8, 12, 14, 15]. Throughout this paper $\mathbb{N}$ will denote the set of natural numbers. Let ψ be a nonempty set. Throughout this paper, for a function $\omega : (0,\infty)\times\Omega\times\Omega \to [0,\infty)$, we write $\omega_\lambda(u,v) = \omega(\lambda,u,v)$ for all $\lambda > 0$ and $u, v \in \Omega$.

Definition 1.1. [4] Let Ω be a nonempty set. A function $\omega : (0,\infty)\times\Omega\times\Omega \to [0,\infty)$ is said to be a metric modular on Ω if it satisfies, for all $u, v, w \in \Omega$, the following condition:

(1) $\omega_\lambda(u,v) = 0$ for all $\lambda > 0$ if and only if $u = v$,
(2) $\omega_\lambda(u,v) = \omega_\lambda(v,u)$ for all $\lambda > 0$,
(3) $\omega_{\lambda+\mu}(u,v) \le \omega_\lambda(u,w) + \omega_\mu(w,v)$ for all $\lambda, \mu > 0$.

If instead of (1) we have only the condition (1') $\omega_\lambda(u,u) = 0$ for all, $u \in \psi, \lambda > 0$ then ω is said to be a pseudo modular (metric) on Ω. An important property of the (metric) pseudo modular on set Ω is that the mapping $\lambda \mapsto \omega_\lambda(u,v)$ is non increasing for all $u, v \in \Omega$.

Definition 1.2. [4] Let ω is a pseudo modular on Ω. Fixed $u_0 \in \Omega$. The set $\Omega_\omega = \Omega_\omega(u_0) = \{u \in \Omega : \omega_\lambda(u_\mu u_0) \to 0 \text{ as } \lambda \to +\infty\}$ is said to be a modular metric space (around u_0).

Definition 1.3. [4] Let Ω_ω be a modular metric space.

(1) The sequence $\{u_\eta\}$ in Ω_ω is said to be ω-convergent to $u \in \Omega_\omega$ if and only if there exists a number $\lambda > 0$, possibly depending on (u_η) and u, such that $\lim_{\mathrm{n}\to+\infty} \omega_\lambda(u_\eta, u) = 0$.

(2) The sequence $\{u_\eta\}$ in Ω_ω is said to be ω-Cauchy if there exists $\lambda > 0$, possibly depending on the sequence, such that $\omega_\lambda(u_m, u_\eta) \to 0$ as $m, \eta \to +\infty$.

(3) A subset H of Ω_ω is said to be ω-complete if any ω-Cauchy sequence in H is a convergent sequence and its limit is in H.

Definition 1.4. [5] Let ω be a metric modular on Ω and Ω_ω be a modular metric space induced by ω. If Ω_ω is a ω complete modular metric space and $\mathcal{T} : \Omega_\omega \to \Omega_\omega$ be an arbitrary mapping T is called a contraction if for each $u, v \in \Omega_\omega$ and for all

$\lambda > 0$ there exists $0 \le \sigma < 1$ such that

$$\omega_\lambda(\mathcal{T}u, \mathcal{T}v) \le \sigma\omega_\lambda(u, v).$$

Now we introduce various type of minimal mappings in modular metric spaces as follow:

Definition 1.5. Two self-maps f and g of a set Ω are called
(i) ω-commutative if $fgu = gfu$ for all $u \in \Omega$.
(ii) ω-weakly commuting if $\omega_\lambda(fgu, gfu) \le \omega_\lambda(fu, gu)$ for each u in Ω.
(iii) ω-compatible if $\lim_{n\to+\infty} \omega_\lambda(fgu_n, gfu_n) = 0$, whenever $\{u_n\}$ is a sequence in Ω such that $\lim_{n\to+\infty} fu_n = \lim_{n\to+\infty} gu_n = t$ for some t $\in \Omega$ and $\lambda > 0$.
(iv) ω-weakly compatible pair if they commute at coincidence points that is, if $fu = gu$ implies $fgu = gfu$.

Remark 1.1. *Clearly, Ω-commuting maps are Ω-weakly commuting and Ω-weakly commuting maps are Ω-compatible.*

Example 1.6. Let $\Omega_\omega = [-\infty, +\infty)$ and $\omega_\lambda(u, v) = (1/\lambda)|u - v|$. Define f, g : $\Omega_\omega \to \Omega_\omega$ as $f(u) = u^2$ and $g(u) = u$. Then take $u_n = \frac{1}{n}, n = 1, 2, \ldots$ we have, $\lim_{n\to+\infty} \frac{1}{n^2} = \lim_{n\to+\infty} \frac{1}{n} = 0$ and $\lim_{n\to+\infty} \omega_\lambda\left(\frac{1}{n^2}, \frac{1}{n^2}\right) = 0$ for $\lambda > 0$. Then (f, g) is ω-compatible at $u = 0$.

Example 1.7. Let $\Omega = [0, 2]$ be equipped with the modular metric space $\omega_\lambda(u, v) = (1/\lambda)|u - v|$. Define $f, g : [0, 2] \to [0, 2]$ by $f(u) = \frac{u^2}{16}$ and $g(u) = \frac{u}{4}$. Then (f, g) is weakly ω-compatible at $u = 0$.

Lemma 1.2. *If the pair (f, g) of self-maps on the Modular metric space (Ω_λ, ω) is ω-compatible, then it is weakly ω-compatible. The converse does not hold.*

Proof. Let $fu = gu$ for some $u \in \Omega$. We have to prove that $fgu = gfu$.
Put $u_n = u$ for every $n \in \mathbb{N}$. We have fu_n, gu_n implies $fu = gu$ then, since the pair (f, g) is compatible, we have $\omega_\lambda(fgu_n, gfu_n) = \omega_\lambda(fgu, gfu) = 0$.
Hence, $\omega_\lambda(fgu, gfu) = 0$ that is, $fgu = gfu$. □

Example 1.8. Let $\Omega = [0, 2]$ be equipped with the modular metric spaces $\omega_\lambda(u, v) = (1/\lambda)|u - v|$. Define $f, g : [0, 2] \to [0, 2]$ by

$$f(u) = \begin{cases} 2 - u & \text{if} \quad 0 \le u < 1 \\ 2 & \text{if} \quad 1 \le u \le 2 \end{cases} \quad \text{and } g(u) = \begin{cases} 2u & \text{if} \quad 0 \le u < 1 \\ u & \text{if} \quad 1 \le u \le 2, \quad u \ne \frac{4}{3} \\ 2 & \text{if} \quad u = \frac{4}{3} \end{cases}$$

For $n \in \mathbb{N}$ such that $n \ge 4$, put $u_n = \frac{2}{3} + \frac{1}{n} \in \Omega$, we have $fu_n = 2 - \left(\frac{2}{3} + \frac{1}{n}\right) = \frac{4}{3} - \frac{1}{n}$ and $gu_n = 2\left(\frac{2}{3} + \frac{1}{n}\right) = \frac{4}{3} + \frac{2}{n}$. We obtained that $fu_n, gu_n \to \frac{4}{3}$. Now $\omega_\lambda\left(fu_n, \frac{4}{3}\right) =$

$(1/\lambda)\left|\frac{4}{3}-\frac{1}{n}-\frac{4}{3}\right|$ and $\omega_\lambda\left(gu_n,\frac{4}{3}\right)=(1/\lambda)\left|\frac{4}{3}+\frac{2}{n}-\frac{4}{3}\right|$ tends to 0 as n tends to $+\infty$. However,

$$\begin{aligned}\omega_\lambda\left(fgu_n,gfu_n\right)=&(1/\lambda)\left|f\left(\frac{4}{3}+\frac{2}{n}\right)-g\left(\frac{4}{3}-\frac{1}{n}\right)\right|\\ =&=(1/\lambda)\left|\frac{4}{3}-\frac{1}{\frac{4}{3}+\frac{2}{n}}-\frac{4}{3}+\frac{2}{\frac{4}{3}-\frac{1}{n}}\right|\\ =&(1/\lambda)\left|-\frac{1}{\frac{4}{3}+\frac{2}{n}}-\frac{2}{\frac{4}{3}-\frac{1}{n}}\right|\\ =&(1/\lambda)\left|\frac{1}{\frac{4}{3}+\frac{2}{n}}+\frac{2}{\frac{4}{3}-\frac{1}{n}}\right|\\ =&(1/\lambda)\left|\frac{3}{4}+\frac{6}{4}\right|\\ =&(1/\lambda)\left|\frac{9}{4}\right|\end{aligned}$$

does not tends to 0 as n tend to $+\infty$. That is, the pair (f,g) is not compatible. Since $f\frac{4}{3}=g\frac{4}{3}$ and $f2=g2$, we have $fg\frac{4}{3}=gf\frac{4}{3}$ and $fg2=gf2$.

2. Main Results

In 2013, Murthy and Prasad [13] proved the following result:
"Let $\mathcal{T}$ be a self-map of a complete metric space Ω satisfying
$[1+p\omega_1(u,v)]\,\omega_1^2(\mathcal{T}u,\mathcal{T}v)\leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\omega_1^2(u,\mathcal{T}u)\omega_1(v,\mathcal{T}v)+\omega_1(u,\mathcal{T}u)\omega_1^2(v,\mathcal{T}v)\right]\\ \omega_1(u,\mathcal{T}u)\omega_2(u,\mathcal{T}v)\omega_1(v,\mathcal{T}u),\\ \omega_2(u,\mathcal{T}v)\omega_1(v,\mathcal{T}u)\omega_1(v,\mathcal{T}v)\end{array}\right\}+m(u,v)-\emptyset m(u,v).$$

Where, $m(u,v)=\max\left\{\begin{array}{c}\omega_1^2(u,v),\omega_1(u,\mathcal{T}u)\omega_1(v,\mathcal{T}v),\omega_2(u,\mathcal{T}v)\omega_1(v,\mathcal{T}u),\\ \frac{1}{2}\left[\omega_1(u,\mathcal{T}u)\omega_2(u,\mathcal{T}v)+\omega_1(v,\mathcal{T}u)\omega_1(v,\mathcal{T}v)\right]\end{array}\right\}$,

$p\geq 0$ is a real number and $\emptyset:[0,+\infty)\to[0,+\infty)$ is a continuous function with $\emptyset(t)=0$ if and only if $t=0$ and $\emptyset(t)>0$ for each $t>0$. Then $\mathcal{T}$ has a unique fixed point in Ω."
Now we extend and generalized the above result in setting of modular metric spaces as follows:

Theorem 2.1. *Let (Ω_ω,ω) be a complete modular metric space. Let A,B,S and T be self- mappings of Ω into itself satisfying the following conditions:*
(C_1) $T(\Omega)\subseteq A(\Omega), S(\Omega)\subseteq B(\Omega)$
(C_2) *If one of the following conditions is satisfied:*
(i) Either A or S is continuous, the pair (A,S) is ω-compatible, the pair (B,T) is ω-weakly compatible;

(ii) Either B or T is continuous, the pair (B,T) is ω-compatible, the pair (A,S) is ω-weakly compatible.

(C$_3$) $[1+p\omega_1(Au,Bv)]\,\omega_1^2(Su,Tv)\leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\omega_1^2(Au,Su)\omega_1(Bv,Tv)+\omega_1(Au,Su)\omega_1^2(Bv,Tv)\right],\\ \omega_1(Au,Su)\omega_2(Au,Tv)\omega_1(Bv,Su),\\ \omega_2(Au,Tv)\omega_1(Bv,Su)\omega_1(Bv,Tv)\end{array}\right\}+$$

$m(Au,Bv)-\emptyset m(Au,Bv),$

where,

(C$_4$)

$$m(Au,Bv)=\max\left\{\begin{array}{c}\omega_1^2(Au,Bv),\omega_1(Au,Su)\omega_1(Bv,Tv),\omega_2(Au,Tv)\omega_1(Bv,Su),\\ \frac{1}{2}\left[\omega_1(Au,Su)\omega_2(Au,Tv)+\omega_1(Bv,Su)\omega_1(Bv,Tv)\right]\end{array}\right\},$$

$p\geq 0$ is a real number and $\emptyset:[0,+\infty)\to[0,+\infty)$ is a continuous function with $\emptyset(t)=0$ if and only if $t=0$ and $\emptyset(t)>0$ for each $t>0$. Then A,B,S and T have a unique common fixed point in Ω_ω.

Proof. Let u_0 be an arbitrary point in Ω. Choose a point $u_1\in\Omega$ such that $v_0=Su_0=Bu_1$. For the point u_1, we can choose a point $u_2\in\Omega$ such that $v_1=Tu_1=Au_2$ as $T(\Omega)\subseteq A(\Omega)$. Continuing this process, we obtain a sequence $\{v_n\}$ in Ω such that $v_{2n}=Su_{2n}=Bu_{2n+1}$ and $v_{2n+1}=Tu_{2n+1}=Au_{2n+2}$. First, we show that $\{v_n\}$ is a Cauchy sequence in Ω.

There are two cases:

Case 1. If n is even, then from (C_3) by taking $u=u_{2n},v=u_{2n+1}$ in inequality (C_3), and for brevity, we write $\alpha_{2n}=\omega_1(v_{2n},v_{2n+1})$.

First, we prove that $\{\alpha_{2n}\}$ is non increasing sequence and converges to zero.

$$[1+p\omega_1(Au_{2n},Bu_{2n+1})]\,\omega_1^2(Su_{2n},Tu_{2n+1})\leq$$
$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1^2(Au_{2n},Su_{2n})\,\omega_1(Bu_{2n+1},Tu_{2n+1})+\\ \omega_1(Au_{2n},Su_{2n})\,\omega_1^2(Bu_{2n+1},Tu_{2n+1})\end{array}\right]\\ \omega_1(Au_{2n},Su_{2n})\,\omega_2(Au_{2n},Tu_{2n+1})\,\omega_1(Bu_{2n+1},Su_{2n}),\\ \omega_2(Au_{2n},Tu_{2n+1})\,\omega_1(Bu_{2n+1},Su_{2n})\,\omega_1(Bu_{2n+1},Tu_{2n+1})\end{array}\right\}+$$
$$(Au_{2n},Bu_{2n+1})-\emptyset(Au_{2n},Bu_{2n+1}),$$

where, $m(Au_{2n},Bu_{2n+1})$

$$=\max\left\{\begin{array}{c}\omega_1^2(Au_{2n},Bu_{2n+1}),\omega_1(Au_{2n},Su_{2n})\,\omega_1(Bu_{2n+1},Tu_{2n+1}),\\ \omega_2(Au_{2n},Tu_{2n+1})\,\omega_1(Bu_{2n+1},Su_{2n}),\\ \frac{1}{2}\left[\begin{array}{c}\omega_1(Au_{2n},Su_{2n})\,\omega_2(Au_{2n},Tu_{2n+1})+\\ \omega_1(Bu_{2n+1},Su_{2n})\,\omega_1(Bu_{2n+1},Tu_{2n+1})\end{array}\right]\end{array}\right\},$$

$$[1+p\omega_1(v_{2n-1},v_{2n})]\,\omega_1^2(v_{2n},v_{2n+1}) \leq$$
$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1^2(v_{2n-1},v_{2n})\,\omega_1(v_{2n},v_{2n+1})+\\ \omega_1(v_{2n-1},v_{2n})\,\omega_1^2(v_{2n},v_{2n+1})\end{array}\right]\\ \omega_1(v_{2n-1},v_{2n})\,\omega_2(v_{2n-1},v_{2n+1})\,\omega_1(v_{2n},v_{2n})\\ \omega_2(v_{2n-1},v_{2n+1})\,\omega_1(v_{2n},v_{2n})\,\omega_1(v_{2n},v_{2n+1})\end{array}\right\}+$$
$$m(v_{2n-1},v_{2n})-\emptyset m(v_{2n-1},v_{2n}),$$

where, $m(v_{2n-1},v_{2n})=\max\left\{\begin{array}{c}\omega_1^2(v_{2n-1},v_{2n}),\omega_1(v_{2n-1},v_{2n})\,\omega_1(v_{2n},v_{2n+1}),\\ \omega_2(v_{2n-1},v_{2n+1})\,\omega_1(v_{2n},v_{2n}),\\ \frac{1}{2}\left[\begin{array}{c}\omega_1(v_{2n-1},v_{2n})\,\omega_2(v_{2n-1},v_{2n+1})+\\ \omega_1(v_{2n},v_{2n})\,\omega_1(v_{2n},v_{2n+1})\end{array}\right]\end{array}\right\}.$

On using $\alpha_{2n}=\omega_1(v_{2n},v_{2n+1})$ in (C_3), we have

$$[1+p\alpha_{2n-1}]\,\alpha_{2n}^2 \leq p\max\left\{\begin{array}{c}\frac{1}{2}\left[\alpha_{2n-1}^2\alpha_{2n}+\alpha_{2n-1}\alpha_{2n}^2\right],0,\\ 0\end{array}\right\}+$$
$$m(v_{2n-1},v_{2n})-\emptyset m(v_{2n-1},v_{2n}),$$

where,

$m(v_{2n-1},v_{2n})=\max\left\{\alpha_{2n-1}^2,\alpha_{2n-1}\alpha_{2n},0,\frac{1}{2}\left[\alpha_{2n-1}\omega_2(v_{2n-1},v_{2n+1})+0\right]\right\}.$

Now using triangular inequality, we have

$\omega_2(v_{2n-1},v_{2n+1}) \leq \omega_1(v_{2n-1},v_{2n})+\omega_1(v_{2n},v_{2n+1})=\alpha_{2n-1}+\alpha_{2n}$

$m(v_{2n-1},v_{2n})=\max\left\{\alpha_{2n-1}^2,\alpha_{2n-1}\alpha_{2n},0,1\frac{1}{2}\left[\alpha_{2n-1}(\alpha_{2n-1}+\alpha_{2n})\right]\right\}.$

If $\alpha_{2n-1}<\alpha_{2n}$, then after simplification, we get

$$[1+p\alpha_{2n-1}]\,\alpha_{2n}^2 \leq p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\alpha_{2n}^2\alpha_{2n}+\\ \alpha_{2n}\alpha_{2n}^2\end{array}\right],\\ 0,\\ 0\end{array}\right\}+m(v_{2n-1},v_{2n})-\emptyset m(v_{2n-1},v_{2n}),$$

where,

$m(v_{2n-1},v_{2n})=\max\left\{\alpha_{2n}^2,\alpha_{2n}\alpha_{2n},0,\frac{1}{2}\left[\alpha_{2n}(\alpha_{2n}+\alpha_{2n})\right]\right\}$

$=\alpha_{2n}^2\ [1+p\alpha_{2n}]\,\alpha_{2n}^2 \leq p\alpha_{2n}^3+\alpha_{2n}^2-\emptyset(\alpha_{2n}^2)$

$0 \leq \emptyset(\alpha_{2n}^2)$, which is a contradiction. Hence $\alpha_{2n} \leq \alpha_{2n-1.}$ In a similar way, if n is odd, then we can obtain $\alpha_{2n+1} \leq \alpha_{2n}$. Therefore, sequence $\{\alpha_{2n}\}$ is monotone decreasing sequence which is bounded below by 0. So, there exists $r \geq 1$ such that $\alpha_{2n} \to r$ as $n \to +\infty$. Suppose $r>0$, then from inequality (C_3), by putting $u=u_{2n}$ and $v=u_{2n+1}$ in (C_3), we have

$$[1+p\omega_1(Au_{2n},Bu_{2n+1})]\,\omega_1^2(Su_{2n},Tu_{2n+1}) \leq$$
$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1^2(Au_{2n},Su_{2n})\,\omega_1(Bu_{2n+1},Tu_{2n+1})+\\ \omega_1(Au_{2n},Su_{2n})\,\omega_1^2(Bu_{2n+1},Tu_{2n+1})\end{array}\right]\\ \omega_1(Au_{2n},Su_{2n})\,\omega_2(Au_{2n},Tu_{2n+1})\,\omega_1(Bu_{2n+1},Su_{2n}),\\ \omega_2(Au_{2n},Tu_{2n+1})\,\omega_1(Bu_{2n+1},Su_{2n})\,\omega_1(Bu_{2n+1},Tu_{2n+1})\end{array}\right\}+$$
$$m(Au_{2n},Bu_{2n+1})-\emptyset m(Au_{2n},Bu_{2n+1}),$$

where, $m(Au_{2n},Bu_{2n+1})$

$$= \max \left\{ \begin{array}{c} \omega_1^2 (Au_{2n}, Bu_{2n+1}), \omega_1 (Au_{2n}, Su_{2n}) \omega_1 (Bu_{2n+1}, Tu_{2n+1}), \\ \omega_2 (Au_{2n}, Tu_{2n+1}) \omega_1 (Bu_{2n+1}, Su_{2n}), \\ \frac{1}{2} \left[\begin{array}{c} \omega_1 (Au_{2n}, Su_{2n}) \omega_2 (Au_{2n}, Tu_{2n+1}) + \\ \omega_1 (Bu_{2n+1}, Su_{2n}) \omega_1 (Bu_{2n+1}, Tu_{2n+1}) \end{array} \right] \end{array} \right\},$$

$[1 + p\omega_1 (v_{2n-1}, v_{2n})] \omega_1^2 (v_{2n}, v_{2n+1}) \leq$

$$p \max \left\{ \begin{array}{c} \frac{1}{2} \left[\begin{array}{c} {\omega_1}^2 (v_{2n-1}, v_{2n}) \omega_1 (v_{2n}, v_{2n+1}) + \\ \omega_1 (v_{2n-1}, v_{2n}) {\omega_1}^2 (v_{2n}, v_{2n+1}) \end{array} \right], \\ \omega_1 (v_{2n-1}, v_{2n}) \omega_2 (v_{2n-1}, v_{2n+1}) \omega_1 (v_{2n}, v_{2n}), \\ \omega_2 (v_{2n-1}, v_{2n+1}) \omega_1 (v_{2n}, v_{2n}) \omega_1 (v_{2n}, v_{2n+1}) \end{array} \right\} +$$

$m (v_{2n-1}, v_{2n}) - \emptyset m (v_{2n-1}, v_{2n})$,

where, $m (v_{2n-1}, v_{2n}) =$

$$\max \left\{ \begin{array}{c} \omega_1^2 (v_{2n-1}, v_{2n}), \omega_1 (v_{2n-1}, v_{2n}) \omega_1 (v_{2n}, v_{2n+1}), \omega_2 (v_{2n-1}, v_{2n+1}) \omega_1 (v_{2n}, v_{2n}), \\ \frac{1}{2} [\omega_1 (v_{2n-1}, v_{2n}) \omega_2 (v_{2n-1}, v_{2n+1}) + \omega_1 (v_{2n}, v_{2n}) \omega_1 (v_{2n}, v_{2n+1})] \end{array} \right\}$$

$$[1 + p\alpha_{2n-1}] \alpha_{2n}^2 \leq p \max \left\{ \frac{1}{2} \left[\begin{array}{c} \alpha_{2n-1}^2 \alpha_{2n} + \\ \alpha_{2n-1} \alpha_{2n}^2 \end{array} \right], 0, 0 \right\} + m (v_{2n-1}, v_{2n}) - \emptyset m (v_{2n-1}, v_{2n}),$$

where,

$$m (v_{2n-1}, v_{2n}) = \max \left\{ \begin{array}{c} \alpha_{2n-1}^2, \alpha_{2n-1} \alpha_{2n}, 0, \\ \frac{1}{2} [\alpha_{2n-1} (\alpha_{2n-1} + \alpha_{2n})] \end{array} \right\}.$$

Now

$$[1 + pr] r^2 \leq p \max \left\{ \begin{array}{c} \frac{1}{2} [r^3 + r^3] \\ 0, \\ 0 \end{array} \right\} + m (v_{2m}, v_{2m-1}) - \emptyset m (v_{2m}, v_{2m-1}).$$

where,

$$m (v_{2n-1}, v_{2n}) = \max \left\{ \begin{array}{c} r^2, r^2, 0, \\ r^2 \end{array} \right\} = r^2. \text{ So,} [1 + pr] r^2 \leq pr^3 + r^2 - \emptyset (r^2).$$

Then, $\emptyset (r^2) \leq 0$, since r is positive, then by property of $\emptyset$, we get $r = 0$, we conclude that $\lim_{n \to +\infty} \alpha_{2n} = r = 0$. Now we show $\{v_n\}$ to be a Cauchy sequence in Ω.

Suppose we assume that $\{v_n\}$ is not a Cauchy sequence.

For given $\epsilon > 0$, we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$\omega_8 \left(v_m(k), v_{n(k)}\right) \geq \epsilon, \ \omega_{\frac{1}{4}} \left(v_{m(k)}, v_{n(k-1)}\right) < \epsilon \text{ and } n(k) > m(k) > k \tag{2.1}$$

Now $\epsilon \leq \omega_8 \left(v_{m(k)}, v_{n(k)}\right) \leq \omega_2 \left(v_{m(k)}, v_{n(k)}\right) + \omega_1 \left(v_{m(k)}, v_{n(k)}\right)$
$\leq \omega_{\frac{1}{2}} \left(v_{m(k)}, v_{n(k-1)}\right) + \omega_{\frac{1}{2}} \left(v_{n(k)-1}, v_{n(k)}\right) \leq \omega_{\frac{1}{4}} \left(v_{m(k)}, v_{n(k-1)}\right) + \omega_{\frac{1}{2}} \left(v_{n(k)-1}, v_{n(k)}\right)$
$\leq \epsilon + \omega_{\frac{1}{2}} \left(v_{n(k)-1}, v_{n(k)}\right)$.

Letting $k \to +\infty$, we get $\lim_{k \to +\infty} \omega_2 \left(v_{m(k)}, v_{n(k)}\right) = \lim_{k \to +\infty} \omega_1 \left(v_{m(k)}, v_{n(k)}\right) = \epsilon$.

Again using triangular inequality, we have

$$\epsilon \leq \omega_8 \left(v_{m(k)}, v_{n(k)}\right) \leq \omega_4 \left(v_{m(k)}, v_{n(k)}\right) \leq \omega_2 \left(v_{n(k)}, v_{n(k)+1}\right) + \omega_2 \left(v_{m(k)}, v_{n(k)+1}\right). \tag{2.2}$$

We get

$\epsilon - \omega_2 \left(v_{n(k)}, v_{n(k)+1}\right) \leq \omega_2 \left(v_{m(k)}, v_{n(k)+1}\right) \leq \omega_1 \left(v_{m(k)}, v_{n(k)+1}\right) \leq \omega_{\frac{1}{4}} \left(v_{m(k)}, v_{n(k)+1}\right)$

$\leq \omega_{\frac{1}{8}}\left(v_{m(k)}, v_{n(k)}\right)+\omega_{\frac{1}{8}}\left(v_{n(k)}, v_{n(k)+1}\right)$

Taking limits as $k \rightarrow +\infty$, we have

$\lim_{k\rightarrow+\infty} \omega_1\left(v_{m(k)}, v_{n(k)+1}\right)=\lim_{k\rightarrow+\infty} \omega_2\left(v_{m(k)}, v_{n(k)+1}\right)=\epsilon.$ (2.3)

Now from the triangular inequality, we have

$\epsilon \leq \omega_2\left(v_{m(k)}, v_{n(k)}\right) \leq \omega_1\left(v_{m(k)}, v_{m(k)+1}\right)+\omega_1\left(v_{m(k)+1}, v_{n(k)}\right).$

We get

$\epsilon-\omega_1\left(v_{m(k)}, v_{m(k)+1}\right) \leq \omega_1\left(v_{m(k)+1}, v_{n(k)}\right)$

$\leq \omega_{\frac{1}{2}}\left(v_{n(k)}, v_{m(k)-1}\right)+\omega_{\frac{1}{2}}\left(v_{m(k)+1}, v_{m(k)-1}\right)$

$\leq \omega_{\frac{1}{2}}\left(v_{n(k)}, v_{m(k)-1}\right)+\omega_{\frac{1}{4}}\left(v_{m(k)-1}, v_{m(k)}\right)+\omega_{\frac{1}{4}}\left(v_{m(k)}, v_{m(k)+1}\right).$

Letting $k \rightarrow +\infty$, we have $\lim_{k\rightarrow+\infty} \omega_1\left(v_{m(k)+1}, v_{n(k)}\right)=\epsilon.$ (2.4)

Again, from the triangular inequality, we have

$\omega_8\left(v_{m(k)}, v_{n(k)}\right) \leq \omega_4\left(v_{n(k)}, v_{n(k)+1}\right)+\omega_4\left(v_{n(k)+1}, v_{m(k)}\right).$

We get

$\omega_8\left(v_{m(k)}, v_{n(k)}\right) \leq \omega_4\left(v_{n(k)}, v_{n(k)+1}\right)+\omega_2\left(v_{m(k)+1}, v_{m(k)}\right)+$

$\omega_2\left(v_{m(k)+1}, v_{n(k)+1}\right)\ \omega_8\left(v_{m(k)}, v_{n(k)}\right)-\omega_4\left(v_{n(k)}, v_{n(k)+1}\right)-\omega_2\left(v_{m(k)+1}, v_{m(k)}\right)$

$\leq \omega_2\left(v_{m(k)+1}, v_{n(k)+1}\right) \leq \omega_1\left(v_{m(k)+1}, v_{m(k)}\right)+\omega_1\left(v_{n(k)+1}, v_{m(k)}\right).$

Letting $k \rightarrow +\infty$, we have $\lim_{k\rightarrow+\infty} \omega_2\left(v_{m(k)+1}, v_{n(k)+1}\right)=\epsilon.$ (2.5)

Since $\omega_2\left(v_{m(k)+1}, v_{n(k)+1}\right) \leq \omega_1\left(v_{m(k)+1}, v_{n(k)+1}\right) \leq \omega_{\frac{1}{2}}\left(v_{m(k)+1}, v_{m(k)}\right)+$

$\omega_{\frac{1}{2}}\left(v_{m(k)}, v_{n(k)+1}\right)$

$\leq \omega_{\frac{1}{2}}\left(v_{m(k)}, v_{n(k)+1}\right) \leq \omega_{\frac{1}{4}}\left(v_{m(k)}, v_{m(k)-1}\right)+\omega_{\frac{1}{4}}\left(v_{m(k)-1}, v_{n(k)-1}\right)$

$\leq \omega_{\frac{1}{8}}\left(v_{m(k)-1}, v_{n(k)}\right)+\omega_{\frac{1}{8}}\left(v_{n(k)-1}, v_{n(k)}\right).$

Letting $k \rightarrow +\infty$, we have $\lim_{k\rightarrow+\infty} \omega_1\left(v_{m(k)+1}, v_{n(k)+1}\right)=\epsilon.$ (2.6)

On putting $u=u_{m(k)}$ and $v=u_{n(k)}$ in (C_3), we get

$$\left[1+p\omega_1\left(Au_{m(k)}, Bu_{n(k)}\right)\right] \omega_1^2\left(Su_{m(k)}, Tu_{n(k)}\right) \leq$$

$$p \max\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \omega_1^2\left(Au_{m(k)}, Su_{m(k)}\right) \omega_1\left(Bu_{n(k)}, Tu_{n(k)}\right)+ \\ \omega_1\left(Au_{m(k)}, Su_{m(k)}\right) \omega_1^2\left(Bu_{n(k)}, Tu_{n(k)}\right) \end{array}\right], \\ \omega_1\left(Au_{m(k)}, Su_{m(k)}\right) \omega_2\left(Au_{m(k)}, Tu_{n(k)}\right) \omega_1\left(Bu_{n(k)}, Su_{m(k)}\right), \\ \omega_2\left(Au_{m(k)}, Tu_{n(k)}\right) \omega_1\left(Bu_{n(k)}, Su_{m(k)}\right) \omega_1\left(Bu_{n(k)}, Tu_{n(k)}\right) \end{array}\right\}+$$

where,

$m\left(Au_{m(k)}, Bu_{n(k)}\right)=$

$$\max\left\{\begin{array}{c} \omega_1^2\left(Au_{m(k)}, Bu_{n(k)}\right), \omega_1\left(Au_{m(k)}, Su_{m(k)}\right) \omega_1\left(Bu_{n(k)}, Tu_{n(k)}\right) \\ \omega_2\left(Au_{m(k)}, Tu_{n(k)}\right) \omega_1\left(Bu_{n(k)}, Su_{m(k)}\right) \\ \frac{1}{2}\left[\begin{array}{c} \omega_1\left(Au_{m(k)}, Su_{m(k)}\right) \omega_2\left(Au_{m(k)}, Tu_{n(k)}\right) \\ + \\ \omega_1\left(Bu_{n(k)}, Su_{m(k)}\right) \omega_1\left(Bu_{n(k)}, Tu_{n(k)}\right) \end{array}\right] \end{array}\right\}.$$

Now

$\left[1+p\omega_1\left(v_{m(k)-1}, v_{n(k)-1}\right)\right] \omega_1^2\left(v_{m(k)}, v_{n(k)}\right) \leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1^2\left(v_{m(k)-1},v_{m(k)}\right)\omega_1\left(v_{n(k)-1},v_{n(k)}\right)+\\ \omega_1\left(v_{m(k)-1},v_{m(k)}\right)\omega_1^2\left(v_{n(k)-1},v_{n(k)}\right)\end{array}\right]\\ \omega_1\left(v_{m(k)-1},v_{m(k)}\right)\omega_2\left(v_{m(k)-1},v_{n(k)}\right)\omega_1\left(v_{n(k)-1},v_{m(k)}\right)\right),\\ \omega_2\left(v_{m(k)-1},v_{n(k)}\right)\omega_1\left(v_{n(k)-1},v_{m(k)}\right)\omega_1\left(v_{n(k)-1},v_{n(k)}\right)\end{array}\right\}+$$

$m\left(v_{m(k)-1},v_{n(k)-1}\right)-\emptyset m\left(v_{m(k)-1},v_{n(k)-1}\right),$

where,

$$m\left(v_{m(k)-1},v_{n(k)-1}\right)$$

$$=\max\left\{\begin{array}{c}\omega_1^2\left(v_{m(k)-1},v_{n(k)-1}\right),\omega_1\left(v_{m(k)-1},v_{m(k)}\right)\omega_1\left(v_{n(k)-1},v_{n(k)}\right)\\ \omega_2\left(v_{m(k)-1},v_{n(k)}\right)\omega_1\left(v_{n(k)-1},v_{m(k)}\right)\\ \frac{1}{2}\left[\begin{array}{c}\omega_1\left(v_{m(k)-1},v_{m(k)}\right)\omega_2\left(v_{m(k)-1},v_{n(k)}\right)+\\ \omega_1\left(v_{n(k)-1},v_{m(k)}\right)\omega_1\left(v_{n(k)-1},v_{n(k)}\right)\end{array}\right]\end{array}\right\}.$$

Letting $k\to+\infty$ and using (2.1)-(2.6), we get

$[1+p\epsilon]\epsilon^2\leq p\max\left\{\ \frac{1}{2}\left[\ 0+0\ \right],0,0\ \right\}+m\left(v_{m(k)-1},v_{n(k)-1}\right)-\emptyset m\left(v_{m(k)-1},v_{n(k)-1}\right),$

where,

$m\left(v_{m(k)-1},v_{n(k)-1}\right)=\max\left\{\ \epsilon^2,0,\epsilon^2,\frac{1}{2}[0+0]\ \right\}=\epsilon^2.$

Thus $\{v_n\}$ is a Cauchy sequence in Ω. Since Ω is complete, there exists, a point $w\in\Omega$ such that $\lim_{n\to+\infty}v_n=w$. Now we show that w is the fixed point for maps A,B,S and T. It is clear that $\lim_{n\to+\infty}v_{2n}=\lim_{n\to+\infty}Su_{2n}=\lim_{n\to+\infty}Bu_{2n+1}=w$ and $\lim_{n\to+\infty}v_{2n+1}=\lim_{n\to+\infty}Tu_{2n+1}=\lim_{n\to+\infty}Au_{2n+2}=w$.

So, $\lim_{n\to+\infty}Au_{2n+2}=\lim_{n\to+\infty}Tu_{2n+1}=\lim_{n\to\infty}Bu_{2n+1}=\lim_{n\to+\infty}Su_{2n}=w$.

There are two cases arise:

Case 1. Suppose that S is a continuous, then

$\lim_{n\to+\infty}SSu_{2n}=\lim_{n\to+\infty}SBu_{2n+1}=Sw\lim_{n\to+\infty}STu_{2n+1}=\lim_{n\to+\infty}SAu_{2n+2}=Sw.$

Since the pair (A,S) is compatible, therefore, $\lim_{n\to+\infty}\omega_1\left(\mathrm{SA}u_{2n+2},\ \mathrm{A}Su_{2n+2}\right)=\lim_{n\to+\infty}\omega_1\left(\mathrm{Sz},\mathrm{ASu_{2n+2}}\right)=0$. Therefore, $\lim_{n\to+\infty}ASu_{2n+2}=Sw$.

Putting $u=Su_{2n},v=u_{2n+1}$ in (C_3), we get

$[1+p\omega_1\left(ASu_{2n},Bu_{2n+1}\right)]\,\omega_1^2\left(SSu_{2n},Tu_{2n+1}\right)\leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1^2\left(ASu_{2n},SSu_{2n+2}\right)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)+\\ \omega_1\left(ASu_{2n},SSu_{2n}\right)\omega_1^2\left(Bu_{2n+1},Tu_{2n+1}\right)\end{array}\right]\\ \omega_1\left(ASu_{2n},\mathrm{SSu_{2n}}\right)\omega_2\left(ASu_{2n},Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1},SSu_{2n}\right)\\ \omega_2\left(ASu_{2n},Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1},SSu_{2n}\right)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)\end{array}\right\}+$$

$m\left(ASu_{2n},Bu_{2n+1}\right)-\emptyset m\left(ASu_{2n},Bu_{2n+1}\right),$

where,

$$m\left(ASu_{2n},Bu_{2n+1}\right)$$

$$=\max\left\{\begin{array}{c}{\omega_1}^2\left(ASu_{2n},Bu_{2n+1}\right),\omega_1\left(ASu_{2n},SSu_{2n}\right)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)\\ \omega_2\left(ASu_{2n},Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1},SSu_{2n}\right)\\ \frac{1}{2}\left[\begin{array}{c}\omega_1\left(ASu_{2n},SSu_{2n}\right)\omega_2\left(ASu_{2n},Tu_{2n+1}\right)+\\ \omega_1\left(Bu_{2n+1},SSu_{2n}\right)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)\end{array}\right]\end{array}\right\}.$$

Letting $n\to+\infty$, we get

$[1+p\omega_1(\mathrm{Sw},w)]\,\omega_1^2(\mathrm{Sw},w)\leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\omega_1^2(\mathrm{Sw},\mathrm{Sw})\omega_1(w,w)+\omega_1(\mathrm{Sw},\mathrm{Sw})\omega_1^2(w,w)\right.\\ \omega_1(\mathrm{Sw},\mathrm{Sw})\omega_2(\mathrm{Sw},w)\omega_1(w,\mathrm{Sw}),\\ \omega_2(\mathrm{Sw},w)\omega_1(w,\mathrm{Sw})\omega_1(w,w)\end{array}\right\}+m(\mathrm{Sw},w)-\emptyset m(\mathrm{Sw},w),$$

where,

$$m(Sw,w)=\max\left\{\begin{array}{c}\omega_1^2(Sw,w),\omega_1(Sw,Sw)\omega_1(w,w),\omega_2(Sw,w)\omega_1(w,Sw)\\ \frac{1}{2}\left[\omega_1(Sw,Sw)\omega_2(Sw,w)+\omega_1(w,Sw)\omega_1(w,w)\right]\end{array}\right\}.$$

That is, $m(Sw,w)=\max\left\{\begin{array}{c}\omega_1^2(Sw,w),0,\omega_2(Sw,w)\omega_1(w,Sw),\\ \frac{1}{2}[0+0]\end{array}\right\}=\omega_1^2(Sw,w).$

Now

$[1+p\omega_1(\mathrm{Sz},w)]\,\omega_1^2(\mathrm{Sz},w)\leq\omega_1^2(Sw,w)-\emptyset\left(\omega_1^2(Sw,w)\right)$

$p\omega_1^3(Sw,w)\leq-\emptyset\left(\omega_1^2(Sw,w)\right).$

Therefore, $Sw=w$.

Now putting $u=w,v=u_{2n+1}$ in (C_3) and using $Sw=w$, we have

$[1+p\omega_1\left(Aw,Bu_{2n+1}\right)]\,\omega_1^2\left(Sw,Tu_{2n+1}\right)\leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1^2(Aw,Sw)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)+\\ \omega_1(Aw,Sw)\omega_1^2\left(Bu_{2n+1},Tu_{2n+1}\right)\end{array}\right]\\ \omega_1(Aw,Sw)\omega_2\left(Aw,Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1},Sw\right)\\ \omega_2\left(Aw,Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1},Sw\right)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)\end{array}\right\}+m\left(Aw,Bu_{2n+1}\right)-$$

$\emptyset m\left(Aw,Bu_{2n+1}\right),$

where,

$$m\left(Aw,Bu_{2n+1}\right)$$
$$=\max\left\{\begin{array}{c}\omega_1^2\left(Aw,Bu_{2n+1}\right),\omega_1(Aw,Sw)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)\\ ,\omega_2\left(Aw,Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1},Sw\right),\\ \frac{1}{2}\left[\omega_1(Aw,Sw)\omega_2\left(Aw,Tu_{2n+1}\right)+\omega_1\left(Bu_{2n+1},Sw\right)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)\right]\end{array}\right\}.$$

Now we have,

$[1+p\omega_1(Aw,w)]\,\omega_1^2(w,w)\leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\ \omega_1^2(Aw,w)\omega_1(w,w)+\omega_1(Aw,w)\omega_1^2(w,w)\ \right],\\ \omega_1(Aw,w)\omega_2(Aw,w)\omega_1(w,w),\\ \omega_2(Aw,w)\omega_1(w,w)\omega_1(w,w)\end{array}\right\}+m(Aw,w)-\emptyset m\left(Aw,w\right),$$

where,

$$m(Aw,w)=\max\left\{\begin{array}{c}\omega_1^2(Aw,w),\omega_1(Aw,w)\omega_1(w,w),\omega_2(Aw,w)\omega_1(w,w),\\ \frac{1}{2}\left[\omega_1(Aw,w)\omega_2(Aw,w)+\omega_1(w,w)\omega_1(w,w)\right]\end{array}\right\}.$$

So, $0\leq 0+m(Aw,w)-\emptyset m(Aw,w)$,

where, $m(Aw,w)=\max\left\{\begin{array}{c}\omega_1^2(Aw,w),0,0,\\ \frac{1}{2}\left[\omega_1(Aw,w)\omega_2(Aw,w)\right]\end{array}\right\}=\omega_1^2(Aw,w).$

This implies that $0\leq\omega_1^2(Aw,w)-\emptyset\left(\omega_1^2(Aw,w)\right)$.

Therefore, $Aw=w$.

On the other hand, since $w=Sw\in S(\Omega)\subseteq B(\Omega)$ there exists $w^*\in\Omega$ such that $w=Sw=Bw^*$, then $w=Sw=Aw=Bw^*$.

To prove that $Tw^*=w$.

This implies that $0 \leq \omega_1^2(Aw,w) - \emptyset\left(\omega_1^2(Aw,w)\right)$.
Therefore $Aw = w$.
On the other hand, since $w = Sw \in S(\Omega) \subseteq B(\Omega)$ there exists $w^* \in \Omega$ such that $w = Sw = Bw^*$ then $w = Sw = Aw = Bw^*$.
To prove that $Tw^* = w$.

$$[1 + p\omega_1(Aw, Bw^*)]\,\omega_1^2(Sw, Tw^*) \leq$$

$$p\max\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c}\omega_1^2(Aw,Sw)\omega_1(Bw^*,Tw^*)+\\ \omega_1(Aw,Sw)\omega_1^2(Bw^*,Tw^*)\end{array}\right] \\ \omega_1(Aw,Sw)\omega_2(Aw,Tw^*)\,\omega_1(Bw^*,Sw) \\ \omega_2(Aw,Tw^*)\,\omega_1(Bw^*,Sw)\,\omega_1(Bw^*,Tw^*)\end{array}\right\} + m(Aw,Bw^*) - \emptyset m(Aw,Bw^*),$$

where,

$$m(Aw,Bw^*) = \max\left\{\begin{array}{c}\omega_1^2(Aw,Bw^*),\omega_1(Aw,Sw)\omega_1(Bw^*,Tw^*),\omega_2(Aw,Tw^*) \\ \omega_1(Bw^*,Sw), \\ \frac{1}{2}\left[\begin{array}{c}\omega_1(Aw,Sw)\omega_2(Aw,Tw^*)+\\ \omega_1(Bw^*,Sw)\,\omega_1(Bw^*,Tw^*)\end{array}\right]\end{array}\right\}.$$

Now we have

$$[1 + p\omega_1(w,w)]\,\omega_1^2(w,Tw^*) \leq$$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\omega_1^2(w,w)\omega_1(w,Tw^*) + \omega_1(w,w)\omega_1^2(w,Tw^*)\right], \\ \omega_1(w,w)\omega_2(w,Tw^*)\,\omega_1(w,w), \\ \omega_2(w,Tw^*)\,\omega_1(w,w)\omega_1(w,Tw^*)\end{array}\right\} + m(w,w) - \emptyset m(w,w)$$

where, $m(w,w) = \max\left\{\begin{array}{c}\omega_1^2(w,w),\omega_1(w,w)\omega_1(w,Tw^*),\omega_2(w,Tw^*)\,\omega_1(w,w), \\ \frac{1}{2}\left[\omega_1(w,w)\omega_2(w,Tw^*) + \omega_1(w,w)\omega_1(w,Tw^*)\right]\end{array}\right\}$

$\omega_1^2(w,Tw^*) \leq p\max\left\{\ \frac{1}{2}[0+0],0,0\ \right\} + m(w,w) - \emptyset m(w,w)$,
where, $m(w,w) = \max\left\{\ 0,0,0,\frac{1}{2}[0+0]\ \right\} = 0$.
We get $\omega_1^2(w,Tw^*) \leq 0$.
This implies that $w = Tw^*$. Therefore, $Tw^* = w = Bw^*$.
Since the pair (B,T) is weakly compatible, so $Tw = TBw^* = BTw^* = Bw$.
Now we prove that $Bw = w$. For this putting $u = w, v = w$ in (C_3), we have

$$[1 + p\omega_1(Aw,Bw)]\,\omega_1^2(Sw,Tw) \leq$$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\omega_1^2(Aw,Sw)\omega_1(Bw,Tw) + \omega_1(Aw,Sw)\omega_1^2(Bw,Tw)\right], \\ \omega_1(Aw,Sw)\omega_2(Aw,Tw)\omega_1(Bw,Sw), \\ \omega_2(Aw,Tw)\omega_1(Bw,Sw)\omega_1(Bw,Tw)\end{array}\right\} +$$

$$m(Aw,Bw) - \emptyset m(Aw,Bw),$$

where,

$$m(Aw,Bw) = \max\left\{\begin{array}{c}\omega_1^2(Aw,Bw), \\ \omega_1(Aw,Sw)\omega_1(Bw,Tw),\omega_2(Aw,Tw)\omega_1(Bw,Sw), \\ \frac{1}{2}\left[\begin{array}{c}\omega_1(Aw,Sw)\omega_2(Aw,Tw)+\\ \omega_1(Bw,Sw)\omega_1(Bw,Tw)\end{array}\right]\end{array}\right\}.$$

Now

$$[1 + p\omega_1(w,Bw)]\,\omega_1^2(w,Bw) \leq$$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\omega_1^2(u,u)\omega_1(Bw,Bw)+\omega_1(w,w)\omega_1^2(Bw,Bw)\right],\\ \omega_1(w,w)\omega_2(w,Bw)\omega_1(Bw,w),\\ \omega_2(w,Bw)\omega_1(Bw,w)\omega_1(Bw,Bw)\end{array}\right\}+$$
$m(w,Bw)-\emptyset m(w,Bw)$,
where,
$$m(w,Bw)=\max\left\{\begin{array}{c}\omega_1^2(w,Bw),\omega_1(w,w)\omega_1(Bw,Bw),\omega_2(w,Bw)\omega_1(Bw,w),\\ \frac{1}{2}\left[\omega_1(w,w)\omega_2(w,Bw)+\omega_1(Bw,w)\omega_1(Bw,Bw)\right]\end{array}\right\}.$$
Now
$[1+p\omega_1(w,Bw)]\,\omega_1^2(w,Bw)\leq$
$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\omega_1^2(u,u)\omega_1(Bw,Bw)+\omega_1(w,w)\omega_1^2(Bw,Bw)\right]\\ \omega_1(w,w)\omega_2(w,Bw)\omega_1(Bw,w),\\ \omega_2(w,Bw)\omega_1(Bw,w)\omega_1(Bw,Bw)\end{array}\right\}+$$
$m(w,Bw)-\emptyset m(w,Bw)$,
where,
$$m(Aw,Bw)=\max\left\{\begin{array}{c}\omega_1^2(w,Bw),\omega_1(w,w)\omega_1(Bw,Bw),\omega_2(w,Bw)\omega_1(Bw,w),\\ \frac{1}{2}\left[\omega_1(w,w)\omega_2(w,Bw)+\omega_1(Bw,w)\omega_1(Bw,Bw)\right]\end{array}\right\}.$$
That is,
$$[1+p\omega_1(w,Bw)]\,\omega_1^2(w,Bw)\leq p\max\left\{\begin{array}{c}\frac{1}{2}[0+0],\\ 0,\\ 0\end{array}\right\}+m(w,Bw)-\emptyset m(w,Bw),$$
where,
$$m(Aw,Bw)=\max\left\{\begin{array}{c}\omega_1^2(w,Bw),0,\omega_2(w,Bw)\omega_1(Bw,w),\\ \frac{1}{2}[0+0]\end{array}\right\}=\omega_1^2(w,Bw).$$
That is,
$[1+p\omega_1(w,Bw)]\,\omega_1^2(w,Bw)\leq\omega_1^2(w,Bw)-\emptyset\left(\omega_1^2(w,Bw)\right)$.
Hence $Bw=w$.
Therefore, $w=Sw=Aw=Tw=Bw$.
This implies w is common fixed point of A,B,S and T.
Case 2. Suppose that A is a continuous,
then $\lim_{n\to+\infty}ASu_{2n}=\lim_{n\to+\infty}A^2u_{2n+2}=Aw$.
Since the pair (A,S) is compatible, therefore, $\lim_{n\to+\infty}\omega_1\left(\mathrm{SA}u_{2n},\mathrm{ASu}_{2n}\right)=\lim_{n\to+\infty}\omega_1\left(\mathrm{Az},\mathrm{SA}u_{2n}\right)=0$. Therefore, $\lim_{n\to+\infty}SAu_{2n}=Aw$.
Putting $u=Au_{2n},v=u_{2n+1}$ in (C_3), we get
$[1+p\omega_1\left(AAu_{2n},Bu_{2n+1}\right)]\,\omega_1^2\left(SAu_{2n},Tu_{2n+1}\right)\leq$
$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1^2\left(AAu_{2n},SAu_{2n}\right)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)+\\ \omega_1\left(AAu_{2n},SAu_{2n}\right)\omega_1^2\left(Bu_{2n+1},Tu_{2n+1}\right)\end{array}\right]\\ \omega_1\left(AAu_{2n},SAu_{2n}\right)\omega_2\left(AAu_{2n},Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1},SAu_{2n}\right),\\ \omega_2\left(AAu_{2n},Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1},SAu_{2n}\right)\omega_1\left(Bu_{2n+1},Tu_{2n+1}\right)\end{array}\right\}+$$
$m\left(AAu_{2n},Bu_{2n+1}\right)-\emptyset m\left(AAu_{2n},Bu_{2n+1}\right)$,
where,

$$m\left(AAu_{2n}, Bu_{2n+1}\right) = \max\left\{\begin{array}{c} {\omega_1}^2\left(AAu_{2n}, Bu_{2n+1}\right), \\ \omega_1\left(AAu_{2n}, SAu_{2n}\right)\omega_1\left(Bu_{2n+1}, Tu_{2n+1}\right), \\ \omega_2\left(AAu_{2n}, Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1}, SAu_{2n}\right), \\ \frac{1}{2}\left[\begin{array}{c} \omega_1\left(AAu_{2n}, SAu_{2n}\right)\omega_2\left(AAu_{2n}, Tu_{2n+1}\right)+ \\ \omega_1\left(Bu_{2n+1}, SAu_{2n}\right)\omega_1\left(Bu_{2n+1}, Tu_{2n+1}\right) \end{array}\right] \end{array}\right\},$$

$$[1+p\omega_1(\mathrm{Az}, w)]\,\omega_1^2(\mathrm{Az}, w) \le p\max\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \omega_1^2(\mathrm{Az},\mathrm{Az})\omega_1(w,w)+ \\ \omega_1(\mathrm{Az},\mathrm{Az})\omega_1^2(w,w) \end{array}\right] \\ \omega_1(\mathrm{Az},\mathrm{Az})\omega_2(\mathrm{Az},w)\omega_1(w,\mathrm{Az}), \\ \omega_2(\mathrm{Az},w)\omega_1(w,\mathrm{Az})\omega_1(w,w) \end{array}\right\}+$$

$m(\mathrm{Az}, w) - \emptyset m(\mathrm{Az}, w)$,
where,

$$m(\mathrm{Az}, w) = \max\left\{\begin{array}{c} \omega_1^2(\mathrm{Az}, w), \omega_1(\mathrm{Az},\mathrm{Az})\omega_1(w,w), \omega_2(\mathrm{Az},w)\omega_1(w,\mathrm{Az}), \\ \frac{1}{2}\left[\omega_1(\mathrm{Az},\mathrm{Az})\omega_2(\mathrm{Az},w)+\omega_1(w,\mathrm{Az})\omega_1(w,w)\right] \end{array}\right\}.$$

That is, $m(Aw, w) = \max\left\{\ \omega_1^2(\mathrm{Az}, w), 0, \omega_2(\mathrm{Az}, w)\omega_1(w, \mathrm{Az}),\right\} = \omega_1^2(Aw, w)$.
Now $[1+p\omega_1(\mathrm{Az}, w)]\,\omega_1^2(\mathrm{Az}, w) \le \omega_1^2(\mathrm{Az}, w) - \emptyset\left(\omega_1^2(\mathrm{Az}, w)\right)$.
Implies that $\omega_1(w, Aw) = 0$. Therefore, $Aw = w$.
Now putting $u = w, \mathrm{y} = u_{2n+1}$ in (C_3) and using $Aw = w$, we have
$[1+p\omega_1\left(Aw, Bu_{2n+1}\right)]\,\omega_1^2\left(Sw, Tu_{2n+1}\right) \le m(Aw, Bv) - \emptyset m(Aw, Bv)$,
where, $m\left(Aw, Bu_{2n+1}\right)$

$$= \max\left\{\begin{array}{c} {\omega_1}^2\left(Aw, Bu_{2n+1}\right), \\ \omega_1(Aw, Sw)\omega_1\left(Bu_{2n+1}, Tu_{2n+1}\right), \omega_2\left(Aw, Tu_{2n+1}\right)\omega_1\left(Bu_{2n+1}, Sw\right), \\ \frac{1}{2}\left[\omega_1(Aw, Sw)\omega_2\left(Aw, Tu_{2n+1}\right)+\omega_1\left(Bu_{2n+1}, Sw\right)\omega_1\left(Bu_{2n+1}, Tu_{2n+1}\right)\right] \end{array}\right\}.$$

Now we have

$$[1+p\omega_1(w, w)]\,\omega_1^2(Sw, w) \le p\max\left\{\begin{array}{c} \frac{1}{2}\left[\omega_1^2(w, Sw)\omega_1(w,w)+\omega_1(w, Sw)\omega_1^2(w,w)\right] \\ \omega_1(w, Sw)\omega_2(w,w)\omega_1(w, Sw), \\ \omega_2(Aw, w)\omega_1(w, Sw)\omega_1(w,w) \end{array}\right\}+$$

$m(w, w) - \emptyset m(w, w)$.
where, $m(w, w) = \max\left\{\begin{array}{c} \omega_1^2(w, w), \omega_1(w, Sw)\omega_1(w, w), \omega_2(w, w)\omega_1(w, Sw) \\ \frac{1}{2}\left[\omega_1(w, Sw)\omega_2(w, w)+\omega_1(w, Sw)\omega_1(w, w)\right] \end{array}\right\}$.
We get $\omega_1(w, Sw) = 0$ that is, $Sw = w$.
Since $w = Sw \in S(\Omega) \subseteq B(\Omega)$ there exists $w^* \in \Omega$ such that $w = Sw = Bw^*$.
To prove that $Tw^* = w$.
Now putting $u = Au_{2n}, v = w^*$ in (C_3), we have
$[1+p\omega_1\left(AAu_{2n}, Bw^*\right)]\,\omega_1^2\left(SAu_{2n}, Tw^*\right) \le$

$$p\max\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \omega_1^2\left(AAu_{2n}, SAu_{2n}\right)\omega_1\left(Bw^*, Tw^*\right)+ \\ \omega_1\left(AAu_{2n}, SAu_{2n}\right)\omega_1^2\left(Bw^*, Tw^*\right) \end{array}\right] \\ \omega_1\left(AAu_{2n}, SAu_{2n}\right)\omega_2\left(AAu_{2n}, Tw^*\right)\omega_1\left(Bw^*, SAu_{2n}\right) \\ \omega_2\left(AAu_{2n}, Tw^*\right)\omega_1\left(Bw^*, SAu_{2n}\right)\omega_1\left(Bw^*, Tw^*\right) \end{array}\right\} + m\left(AAu_{2n}, Bw^*\right) -$$

$\emptyset m\left(AAu_{2n}, Bw^*\right)$,
where,

$$m\left(AAu_{2n},Bw^{*}\right)=\max\left\{\begin{array}{c}\omega_1^2\left(AAu_{2n},Bw^{*}\right),\omega_1\left(AAu_{2n},SAu_{2n}\right)\omega_1\left(Bw^{*},Tw^{*}\right)\\ \omega_2\left(AAu_{2n},Tw^{*}\right)\omega_1\left(Bw^{*},SAu_{2n}\right),\\ \frac{1}{2}\left[\begin{array}{c}\omega_1\left(AAu_{2n},SAu_{2n}\right)\omega_2\left(AAu_{2n},Tw^{*}\right)+]\\ \omega_1\left(Bw^{*},SAu_{2n}\right)\omega_1\left(Bw^{*},Tw^{*}\right)\end{array}\right]\end{array}\right\},$$

$$[1+p\omega_1(w,w)]\,\omega_1^2\left(w,Tw^{*}\right)\leq p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1^2(w,w)\omega_1\left(w,Tw^{*}\right)+\\ \omega_1(w,w)\omega_1^2\left(w,Tw^{*}\right)\end{array}\right]\\ \omega_1(w,w)\omega_2\left(w,Tw^{*}\right)\omega_1(w,w),\\ \omega_2\left(w,Tw^{*}\right)\omega_1(w,w)\omega_1\left(w,Tw^{*}\right)\end{array}\right\}+m(w,w)-$$

$\emptyset m(w,w)$,

where, $m(w,w)=\max\left\{\begin{array}{c}\omega_1^2(w,w),\omega_1(w,w)\omega_1\left(w,Tw^{*}\right),\omega_2\left(w,Tw^{*}\right)\omega_1(w,w)\\ \frac{1}{2}\left[\omega_1(w,w)\omega_2\left(w,Tw^{*}\right)+\omega_1(w,w)\omega_1\left(w,Tw^{*}\right)\right]\end{array}\right\}$.

Implies that $\omega_1\left(w,Tw^{*}\right)=0$. Therefore, $Tw^{*}=w$. Hence $Tw^{*}=\mathrm{z}=Bw^{*}$.

Since the pair (B,T) is weakly compatible so $Tw=TBw^{*}=BTw^{*}=Bw$.

Now we prove that $Tw=w$. Now putting $u=u_{2n},v=w$ in (C_3), we have

$[1+p\omega_1\left(Au_{2n},Bw\right)]\,{\omega_1}^2\left(Su_{2n},Tw\right)\leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}{\omega_1}^2\left(Au_{2n},Su_{2n}\right)\omega_1(Bw,Tw)+\\ \omega_1\left(Au_{2n},Su_{2n}\right)\omega_1^2(Bw,Tww)\end{array}\right]\\ \omega_1\left(Au_{2n},Su_{2n}\right)\omega_2\left(Au_{2n},Tw\right)\omega_1\left(Bw,Su_{2n}\right),\\ \omega_2\left(Au_{2n},Tw\right)\omega_1\left(Bw,Su_{2n}\right)\omega_1(Bw,Tw)\end{array}\right\}+$$

$m\left(Au_{2n},Bw\right)-\emptyset m\left(Au_{2n},Bw\right)$,

where,

$$m\left(Au_{2n},Bw\right)=\max\left\{\begin{array}{c}\omega_1^2\left(Au_{2n},Bw\right),\omega_1\left(Au_{2n},Su_{2n}\right)\omega_1(Bw,Tw)\\ \omega_2\left(Au_{2n},Tw\right)\omega_1\left(Bw,Su_{2n}\right)\\ \frac{1}{2}\left[\begin{array}{c}\omega_1\left(Au_{2n},Su_{2n}\right)\omega_2\left(Au_{2n},Tw\right)+\\ \omega_1\left(Bw,Su_{2n}\right)\omega_1(Bw,Tw)\end{array}\right]\end{array}\right\}.$$

Implies that $[1+p\omega_1(w,Bw)]\,\omega_1^2(w,Tw)\leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\omega_1^2(w,w)\omega_1(Bw,Tw)+\omega_1(w,w)\omega_1^2(Bw,Tw)\right],\\ \omega_1(w,w)\omega_2(w,Tw)\omega_1(Bw,w),\\ \omega_2(w,Tw)\omega_1(Bw,w)\omega_1(Bw,Tw)\end{array}\right\}+$$

$m(w,Bw)-\emptyset m(w,Bw)$,

where,

$$m(w,Bw)=\max\left\{\begin{array}{c}\omega_1^2(w,Bw),\omega_1(w,w)\omega_1(Bw,Tw)\\ \omega_2(w,Tw)\omega_1(Bw,w)\\ \frac{1}{2}\left[\begin{array}{c}\omega_1(w,w)\omega_2(w,Tw)+\\ \omega_1(Bw,w)\omega_1(Bw,Tw)\end{array}\right]\end{array}\right\}.$$

That is,

$[1+p\omega_1(w,Tw)]\,\omega_1^2(w,Tw)\leq p\max\left\{\ \frac{1}{2}[0+0],0,0\ \right\}+m(w,Tw)-\emptyset m(w,Tw)$,

where,

$m(w,Tw)=\max\left\{\ \omega_1^2(w,Tw),0,\omega_2(w,Tw)\omega_1(Tw,w),\right\}=\omega_1^2(w,Tw)$.

Implies that $Tw=w$. Therefore, $w=Tw=Bw$.

On the other hand, since $w=Tw\in T(\Omega)\subseteq A(\Omega)$, there exists $w^{**}\in$ X such that

$w = Tw = Aw^{**}$. Now we prove that $Sw^{**} = z$.
Now putting $u = w^{**}, v = w$ in (C_3), we get $Sw^{**} =$ z $Sw^{**} = w = Aw^{**}$.
Since the pair (A, S) is weakly compatible so $Sw = SAw^{**} = ASw^{**} = Aw$ so $Aw = Sw$. Hence $w = Aw = Bw = Sw = Tw$.
Next, we prove A, B, S and T have unique common fixed point.
Uniqueness can be easily found. Therefore, z is unique common fixed point of A, B, S and T. Finally, if condition (ii) of (C_2) hold, then the conclusion is similar to that above, so we omit it. This completes the proof. □

Theorem 2.2. *Let (Ω_ω, ω) be a complete modular metric space. Let A, B, S and T be self- mappings of Ω into itself satisfying the following conditions (C_1) and following:*
(C_4) the pairs (A, S) and (B, T) are ω-commutative mappings,
(C_5) one of A, B, S and T is continuous,
(C_6) $1 + p\omega_1(Au, Bv)]\,{\omega_1}^2\left(S^pu, T^qv\right) \leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[{\omega_1}^2\left(Au, S^pu\right)\omega_1\left(Bv, T^qv\right) + \omega_1\left(Au, S^pu\right){\omega_1}^2\left(Bv, T^qv\right)\right],\\ \omega_1\left(Au, S^pu\right)\omega_2\left(Au, T^qv\right)\omega_1\left(Bv, S^pu\right),\\ \omega_2\left(Au, T^qv\right)\omega_1\left(Bv, S^pu\right)\omega_1\left(Bv, T^qv\right) +\end{array}\right\} +$$

$m(Au, Bv) - \emptyset m(Au, Bv),$
where,

$$m(Au, Bv) = \max\left\{\begin{array}{c}\omega_1^2(Au, Bv), \omega_1\left(Au, S^pu\right)\omega_1\left(Bv, T^qv\right),\\ \omega_2\left(Au, T^qv\right)\omega_1\left(Bv, S^pu\right)\\ \frac{1}{2}\left[\omega_1\left(Au, S^pu\right)\omega_2\left(Au, T^qv\right) + \omega_1\left(Bv, S^pu\right)\omega_1\left(Bv, T^qv\right)\right]\end{array}\right\},$$

$p \geq 0$ is a real number and $\emptyset : [0, +\infty) \to [0, +\infty)$ is a continuous function with $\emptyset(t) = 0$ if and only if $t = 0$ and $\emptyset(t) > 0$ for each $t > 0$ and $p, q \in \mathbb{Z}^+$.
Then A, B, S and T have a unique common fixed point in Ω_ω.

Proof. From $S(\Omega) \subset B(\Omega), T(\Omega) \subset A(\Omega)$, we have
$S^p\Omega \subset S^{p-1}\Omega \subset \cdots \subset S^2\Omega \subset S\Omega \subset B\Omega$
and
$T^q\Omega \subset T^{q-1}\Omega \subset \cdots \subset T^2\Omega \subset T\Omega \subset A\Omega.$
Since the pairs (S, A) and (T, B) are commutative mappings,
$S^pA = S^{p-1}SA = S^{p-1}AS = S^{p-2}(SA)S = S^{p-2}AS^2 = \cdots = AS^p$
and
$T^qB = T^{q-1}TB = T^{q-1}BT = T^{q-2}(TB)T = T^{q-1}BT^2 = \cdots = BT^q.$
That is to say, $S^pA = AS^p$ and $T^qB = BT^q$.
It follows from Remark 1.6 that the pairs (S^p, A) and (T^q, B) are compatible and also weakly compatible. Therefore, by Theorem 2.1, we can find that $S^p, T^q, A,$ and B have a unique common fixed point w. In addition, we prove that $A, B, S,$ and T have a unique common fixed point.
From (C_6), by putting $u = Sw, v = w$, we have
$[1 + p\omega_1(\mathrm{ASz}, Bw)]\,\omega_1^2\left(S^p\mathrm{Sz}, T^qw\right) \leq$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1^2\left(A\mathrm{Sz},S^p\mathrm{Sz}\right)\omega_1\left(Bw,T^qw\right)+\\ \omega_1\left(A\mathrm{Sz},S^p\mathrm{Sz}\right)\omega_1^2\left(Bw,T^qw\right)\end{array}\right]\\ \omega_1\left(A\mathrm{Sz},S^p\mathrm{Sz}\right)\omega_2\left(A\mathrm{Sz},T^qw\right)\omega_1\left(Bw,S^p\mathrm{Sz}\right),\\ \omega_2\left(A\mathrm{Sz},T^qw\right)\omega_1\left(Bw,S^p\mathrm{Sz}\right)\omega_1\left(Bw,T^qw\right)\end{array}\right\}+m(A\mathrm{Sz},Bw)-\emptyset m(A\mathrm{Sz},Bw),$$

where,

$$m(ASz,Bw)=\max\left\{\begin{array}{c}\omega_1^2(ASz,Bw),\omega_1\left(ASz,S^pSz\right)\omega_1\left(Bw,T^qv\right),\\ \omega_2\left(ASz,T^qw\right)\omega_1\left(Bw,S^pSz\right)\\ \frac{1}{2}\left[\begin{array}{c}\omega_1\left(ASz,S^p\mathrm{Sz}\right)\omega_2\left(ASz,T^qw\right)+\\ \omega_1\left(Bw,S^pSz\right)\omega_1\left(Bw,T^qw\right)\end{array}\right]\end{array}\right\},$$

$$[1+p\omega_1(\mathrm{Sz},w)]\,\omega_1^2(\mathrm{Sz},w)\leq$$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\omega_1^2(\mathrm{Sz},\mathrm{Sz})\omega_1(Bw,w)+\omega_1(\mathrm{Sz},\mathrm{Sz})\omega_1^2(Bw,w)\right],\\ \omega_1(\mathrm{Sz},\mathrm{Sz})\omega_2(\mathrm{Sz},w)\omega_1(w,\mathrm{Sz})\\ \omega_2(\mathrm{Sz},w)\omega_1(w,\mathrm{Sz})\omega_1(w,w)\end{array}\right\}+m(\mathrm{Sz},w)-\emptyset m(\mathrm{Sz},w),$$

where,

$$m(\mathrm{Sz},w)=\max\left\{\begin{array}{c}\omega_1^2(\mathrm{Sz},w),\omega_1(\mathrm{Sz},\mathrm{Sz})\omega_1(w,w),\omega_2(\mathrm{Sz},w)\omega_1(w,\mathrm{Sz})\\ \frac{1}{2}\left[\omega_1(\mathrm{Sz},\mathrm{Sz})\omega_2(\mathrm{Sz},w)+\omega_1(w,\mathrm{Sz})\omega_1(w,w)\right]\end{array}\right\}=\omega_1^2(\mathrm{Sz},w)$$

$$[1+p\omega_1(Sw,w)]\,\omega_1^2(Sw,w)\leq\omega_1^2(\mathrm{Sz},w)-\emptyset\left(\omega_1^2(\mathrm{Sz},w)\right).$$

Implies that $\omega_1(Sw,w)=0$ i.e, $Sw=w$.

From (C_6), putting $u=w,v=Tz$, we have

$$[1+p\omega_1(Aw,B\mathrm{Tz})]\,\omega_1^2\left(S^pw,T^q\mathrm{Tz}\right)\leq$$

$$p\max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}{\omega_1}^2\left(Aw,S^pw\right)\omega_1\left(B\mathrm{Tz},T^q\mathrm{Tz}\right)+\\ \omega_1\left(Aw,S^pw\right){\omega_1}^2\left(B\mathrm{Tz},T^q\mathrm{Tz}\right)\end{array}\right],\\ \omega_1\left(Aw,S^pw\right)\omega_2\left(Aw,T^q\mathrm{Tz}\right)\omega_1\left(B\mathrm{Tz},S^pw\right),\\ \omega_2\left(Aw,T^q\mathrm{Tz}\right)\omega_1\left(B\mathrm{Tz},S^pw\right)\omega_1\left(B\mathrm{Tz},T^q\mathrm{Tz}\right)\end{array}\right\}+$$

$$m(Aw,B\mathrm{Tz})-\emptyset m(Aw,B\mathrm{Tz}),$$

where, $$m(Aw,BTz)=\max\left\{\begin{array}{c}\omega_1^2(Aw,B\mathrm{Tz}),\\ \omega_1\left(Aw,S^pw\right)\omega_1\left(B\mathrm{Tz},T^qTz\right),\\ \omega_2\left(Aw,T^q\mathrm{Tz}\right)\omega_1\left(B\mathrm{Tz},S^pw\right),\\ \frac{1}{2}\left[\begin{array}{c}\omega_1\left(Aw,S^pw\right)\omega_2\left(Aw,T^q\mathrm{Tz}\right)+\\ \omega_1\left(B\mathrm{Tz},S^pw\right)\omega_1\left(B\mathrm{Tz},T^q\mathrm{Tz}\right)\end{array}\right]\end{array}\right\}.$$

This implies that $\omega_1(w,Tw)=0$, i.e., $Tw=w$. Therefore, we obtain $Sw=Tw$ $=Aw=Bw=w$, so w is a common fixed point of A,B,S and T.

Finally, we prove that A,B,S and T have a unique common fixed point.

Suppose that $p\in\Omega$ is also a common fixed point of A,B,S and T, then putting $u=w,v=p$ in (C_6), we get $\omega_1(w,p)=0$, and so $w=p$. Therefore, maps A,B,S and T has a unique common fixed point. □

Example 2.1. Let $\Omega=[0,2]$ be equipped with the modular metric space $\omega_\lambda(u,v)=\frac{|\mathrm{x}-\mathrm{y}|}{\lambda}$. Let A,B,S and T be four self-mappings defined by

$Su=\frac{7}{6}$, for all $u\in[0,2]$,

$$Tu=\begin{cases}\frac{3}{2}, & u\in[0,1]\\ \\ \frac{7}{6}, & u\in(1,2]\end{cases},\quad Au=\begin{cases}1, & u\in[0,1]\\ \\ \frac{7}{6}, & u\in(1,2)\\ \\ \frac{3}{2}, & u=2\end{cases},\quad Bu=\begin{cases}\frac{1}{6}, & u\in[0,1]\\ \\ \frac{7}{6}, & u\in(1,2).\\ \\ 1, & u=2\end{cases}$$

Clearly, we get $S(\Omega)\subset B(\Omega)$ and $T(\Omega)\subset A(\Omega)$. Note that A,B and T are not continuous mappings, and S is continuous in Ω. Clearly, (A,S) and (B, T) are ω-commutative mappings. So all the conditions of Theorem 2.1 are satisfied.
Moreover, $\frac{7}{6}$ is the unique common fixed point for all of the mappings A,B,S and T.

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