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Homotopy perturbation method with the help of Adomian decomposition method for nonlinear problems

Soumeyeh Khaleghizadeh

Abstract. This paper concerns He's Homotopy Perturbation Method (HPM) which has been applied to solve some nonlinear differential equations. In HPM, at first, we construct a homotopy that satisfies an equation which is called the perturbation equation. Moreover, in this method, the solution is considered as power series in p. By substituting this series into an equation and equating the coefficient of the terms with identical powers of p, the researcher obtained a set of equations. These equations can be solved in various methods. Here Adomian decomposition method (ADM) is employed for solving equations, obtained from the homotopy perturbation method.

1. Introduction

There is a compelling evident that nonlinear differential equations can be used to describe many physical problems. In recent years, some promising approximate analytical solutions are proposed, such as Exp-function method[1, 2], Adomian decomposition method [3, 4], Variational iteration method [5, 6] and Homotopy perturbation method [7, 8]. It is noteworthy that based on Existance Theoreom for an equation with a parameter, HPM is an effective method for both linear and nonlinear equations. Employing homotopy technique in topology, a homotopy has been constructed with an embedding parameter $p \in [0, 1]$ which is considered as a small parameter.

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In homotopy perturbation method addressed in [7, 8], we obtain some differential equations. There are various methods for solving them. In the following, we employ Adomian decomposition method for solving equations those are obtained from homotopy perturbation method. The results reveal that HPM and ADM are powerful methods for solving nonlinear problems.

2. Examples

To illustrate the effectiveness of both proposed methods, we have chosen two nonlinear differential equations as shown in the following examples. For the first example, the nonlinear dynamic of a particle on a rotating parabola is of concern. Additionally, the governing equation of motion and initial conditions are introduced in Example 1. The second example is associated with the important class of dynamical systems comprising the second order vibration equation.

Example 2.1. Consider the nonlinear differential equation [8].

$$\frac{d^2u}{dt^2} + \omega^2 u + 4q^2u^2\frac{d^2u}{dt^2} + 4q^2u\left(\frac{du}{dt}\right)^2 = 0, \qquad t \in \Omega. \tag{1}$$

With the initial conditions $u(0) = A$ and $u'(0) = 0$.

Where ω and q are known constants.

We construct a homotopy $v(r,p) : \Omega \times [0,1] \to \mathfrak{R}$ which satisfies:

$$L(v) - L(u_0) + pL(u_0) + p\left[4q^2v^2\frac{d^2v}{dt^2} + 4q^2v\left(\frac{dv}{dt}\right)^2\right] = 0, \tag{2}$$

where $L(u) = \frac{d^2u}{dt^2} + \omega^2 u$ and $p \in [0,1]$ is an embedding parameter.

Let's initial approximation of Eq. (1) the following form:

$$u_0(t) = A\cos\alpha\omega t. \tag{3}$$

Where $\alpha(q)$ is a non-zero unknown constant with $\alpha(0) = 1$.

According to homotopy perturbation method the approximation solution of Eq. (2) has the following form:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots \tag{4}$$

Substituting Eq. (4) into Eq. (2), and equating the terms with the identical powers of p we have:

$$L(v_0) - L(u_0) = 0, \qquad v_0(0) = A, \qquad v_0'(0) = 0, \tag{5}$$

$$L(v_1) + L(u_0) + 4q^2v_0^2\frac{d^2v_0}{dt^2} + 4q^2v_0\left(\frac{dv_0}{dt}\right)^2 = 0, \qquad v_1'(0) = v_1(0) = 0. \tag{6}$$

We can consider:

$$v_0 = u_0 = A\cos\alpha t. \tag{7}$$

Substituting Eq. (7) into Eq. (6) results in:

$$L(v_1)+(-\alpha^2+1-2q^2\alpha^2A^2)\omega^2A\cos\alpha\omega t-2q^2\alpha^2\omega^2A^3\cos 3\alpha\omega t=0, \qquad v_1'(0)=v_1. \tag{8}$$

The constant α can be identified by various methods, such as the method of weighted residuals (Least-square method, Collocation method, Galerkin method). Herein the Galerkin method is used to determine the unknown constant, setting:

$$\int_0^{\frac{\pi}{\alpha\omega}} \sin\alpha\omega t\{(-\alpha^2+1-2q^2\alpha^2A^2)\omega^2A\cos\alpha\omega t-2q^2\alpha^2\omega^2A^3\cos 3\alpha\omega t\}dt=0. \tag{9}$$

Leads to:

$$\alpha = 1\Big/\sqrt{1+2q^2A^2}\,. \tag{10}$$

As a result, Eq. (8) reduces to:

$$L(v_1)-2q^2\alpha^2\omega^2A^3\cos 3\alpha\omega t=0, \qquad v_1'(0)=v_1(0)=0. \tag{11}$$

With α defined in Eq. (10).

Let:

$$B=2q^2\alpha^2\omega^2A^3. \tag{12}$$

Therefore Eq.(11) reduces to the following equation:

$$L(v_1)-B\cos 3\alpha\omega t=0, \qquad v_1'(0)=v_1(0)=0. \tag{13}$$

According to the definition of L we obtain:

$$\frac{d^2v_1}{dt^2}+\omega^2v_1-B\cos 3\alpha\omega t=0, \qquad v_1'(0)=v_1(0)=0. \tag{14}$$

By two successive integration of both sides of Eq. (14) from 0 to t we obtain:

$$v_1(t)=-\omega^2\int_0^t (t-x)\left(v_1(x)-\frac{B}{(3\alpha\omega)^2}\cos 3\alpha\omega t+\frac{B}{(3\alpha\omega)^2}\right)dx. \tag{15}$$

In order to solve Eq. (15) we can use Adomian decomposition method:

$$v_{01}(t)=\frac{B}{(3\alpha\omega)^2}(1-\cos 3\alpha\omega t), \tag{16}$$

$$v_{11}(t)=\frac{-\omega^2B}{(3\alpha\omega)^2}\left[\frac{t^2}{2!}+\frac{\cos 3\alpha\omega t-1}{(3\alpha\omega)^2}\right], \tag{17}$$

$$v_{21}(t)=\frac{\omega^4B}{(3\alpha\omega)^2}\left[\frac{t^4}{4!}-\frac{t^2}{2(3\alpha\omega)^2}+\frac{1-\cos 3\alpha\omega t}{(3\alpha\omega)^4}\right], \tag{18}$$

$$v_{31}(t)=\frac{-\omega^6B}{(3\alpha\omega)^2}\left[\frac{t^6}{6!}-\frac{t^4}{4!(3\alpha\omega)^2}+\frac{t^2}{2!(3\alpha\omega)^4}+\frac{\cos 3\alpha\omega t-1}{(3\alpha\omega)^6}\right]. \tag{19}$$

$$\vdots$$

Substituting B from (12) into Eqs. (16)-(19) the following relations will be obtained:

$$v_{01}(t) = \frac{2}{9}q^2A^3(1-\cos 3\alpha\omega t), \tag{20}$$

$$v_{11}(t) = \frac{2}{9}\omega^2q^2A^3\left(-\frac{t^2}{2!}+\frac{1}{(3\alpha\omega)^2}-\frac{1}{(3\alpha\omega)^2}\cos 3\alpha\omega t\right), \tag{21}$$

$$v_{21}(t) = \frac{2}{9}\omega^2q^2A^3\left(\frac{t^4}{4!}-\frac{1}{(3\alpha\omega)^2}\cdot\frac{t^2}{2!}+\frac{1}{(3\alpha\omega)^4}-\frac{1}{(3\alpha\omega)^4}\cos 3\alpha\omega t\right), \tag{22}$$

$$v_{31}(t) = \frac{2}{9}\omega^2q^2A^3\left(-\frac{t^6}{6!}+\frac{1}{(3\alpha\omega)^2}\cdot\frac{t^4}{4!}-\frac{1}{(3\alpha\omega)^4}\cdot\frac{t^2}{2!}+\frac{1}{(3\alpha\omega)^6}-\frac{1}{(3\alpha\omega)^6}\cos 3\alpha\omega t\right), \tag{23}$$

$$\vdots$$

In this procedure four components of the decomposition series are obtained.

Continuing the expansion to the last form, series converges to the solution as follows:

$$\begin{aligned}
v_1(t) &= v_{01}(t)+v_{11}(t)+v_{21}(t)+v_{31}(t)+\cdots \\
&= \frac{2}{9}q^2A^3-\frac{2}{9}q^2A^3\cos 3\alpha\omega t+\frac{2}{9}\omega^2q^2A^3\left[-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots+\frac{1}{(3\alpha\omega)^2}\right. \\
&\times\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots\right)+\frac{1}{(3\alpha\omega)^4}\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots\right) \\
&\left.+\cdots-\cos 3\alpha\omega t\left(\frac{1}{(3\alpha\omega)^2}+\frac{1}{(3\alpha\omega)^4}+\frac{1}{(3\alpha\omega)^6}+\cdots\right)\right] \\
&= \frac{2}{9}q^2A^3-\frac{2}{9}q^2A^3\cos 3\alpha\omega t+\frac{2}{9}\omega^2q^2A^3\left[\cos t-1+\frac{\cos t}{9\alpha^2\omega^2-1}-\frac{\cos 3\alpha\omega t}{9\alpha^2\omega^2-1}\right],
\end{aligned} \tag{24}$$

if $\omega = 1$:

$$\begin{aligned}
v_1(t) &= \cos 3\alpha t\left[\frac{-\frac{2}{9}q^2A^3-\frac{2}{9}q^2A^3(9\alpha^2-1)}{9\alpha^2-1}\right]+\cos t\left[\frac{\frac{2}{9}\omega^2q^2A^3+\frac{2}{9}q^2A^3(9\alpha^2-1)}{9\alpha^2-1}\right] \\
&= -\frac{2q^2A^3\alpha^2}{9\alpha^2-1}\left[\cos 3\alpha t-\cos t\right].
\end{aligned} \tag{25}$$

Thus the first approximation of Eq. (1) will be derived as:

$$u_1(t) = v_0(t)+v_1(t) = A\cos\alpha\omega t-\frac{2q^2\alpha^2A^3}{9\alpha^2-1}(\cos 3\alpha t-\cos t). \tag{26}$$

Which is exactly the same as those obtain by variational iteration method [8].

Example 2.2. consider the following nonlinear equation:

$$\begin{aligned} &\frac{d^2u}{dt^2} + \frac{\omega^2 u}{1+\varepsilon u^2} = 0, \qquad t \in \Omega, \\ &u(0) = Au'(0) = 0, \qquad t \in \Omega. \end{aligned} \tag{27}$$

Let's construct a homotopy $v(r,p) : \Omega \times [0,1] \to \Re$ as the following:

$$(1-p)[L(v) - L(u_0)] + p\left[(1+\varepsilon v^2)v'' + \omega^2 v\right] = 0. \tag{28}$$

Where $L(u) = \frac{d^2u}{dt^2}$ and u_0 is an initial approximation that satisfies in the initial conditions.

Assuming that initial approximation of Eq. (27) be taken as the follows:

$$u_0(t) = A\cos\alpha\omega t. \tag{29}$$

Where α is a non-zero unknown constant with $\alpha(0) = 1$.

By the same manipulation in the above example we have:

$$L(v_0) - L(u_0) = 0, \qquad v_0(0) = A, \qquad v_0'(0) = 0, \tag{30}$$

$$L(v_1) - L(v_0) + L(u_0) + (1+\varepsilon v_0^2)v_0'' + \omega^2 v_0 = 0, \qquad v_1'(0) = v_1(0) = 0. \tag{31}$$

Set $v_0 = u_0 = A\cos\alpha\omega t$. The unknown α can be determined by the Galerkin method:

$$\int_0^{\frac{\pi}{\alpha\omega}} \sin\alpha\omega t \left\{\omega^2 A\left(-\alpha^2 - \frac{3\varepsilon A^2}{4}\alpha^2 + 1\right)\cos\alpha\omega t - \alpha^2\omega^2\frac{\varepsilon A^3}{4}\cos 3\alpha\omega t\right\} dt = 0. \tag{32}$$

The unknown α therefore can be identified as:

$$\alpha = 1 \Bigg/ \sqrt{1 + \frac{3\varepsilon A^2}{4}} \tag{33}$$

As a result, from Eq. (31) we obtain:

$$L(v_1) + \omega^2 A\left(-\alpha^2 - \frac{3}{4}\varepsilon A^2\alpha^2 + 1\right)\cos\alpha\omega t - \alpha^2\omega^2\frac{\varepsilon A^3}{4}\cos 3\alpha\omega t = 0, \tag{34}$$

or:

$$v_1'' + \omega^2 v_1 - \alpha^2\omega^2\frac{\varepsilon A^3}{4}\cos 3\alpha\omega t = 0. \tag{35}$$

By two successive integration of both sides of Eq. (35) in $[0,t]$ we obtain:

$$v_1(t) = -\omega^2\int_0^t (t-x)v_1(x)dx - \frac{\varepsilon A^3}{36}\cos 3\alpha\omega t + \frac{\varepsilon A^3}{36}. \tag{36}$$

In order to solve this equation, we have employed Adomian decomposition method:

$$v_{01}(t) = \frac{\varepsilon A^3}{36}(1-\cos 3\alpha\omega t), \tag{37}$$

$$v_{11}(t) = \frac{\varepsilon A^3}{36}\left[-\frac{\omega^2 t^2}{2!} + \frac{\omega^2}{(3\alpha\omega)^2} - \frac{\omega^2}{(3\alpha\omega)^2}\cos 3\alpha\omega t\right], \tag{38}$$

$$v_{21}(t) = \frac{\varepsilon A^3}{36}\left[\frac{\omega^4 t^4}{4!} - \frac{\omega^4 t^2}{2(3\alpha\omega)^2} + \frac{\omega^4}{(3\alpha\omega)^4} - \frac{\omega^4}{(3\alpha\omega)^4}\cos 3\alpha\omega t\right], \tag{39}$$

$$v_{31}(t) = \frac{\varepsilon A^3}{36}\left[-\frac{\omega^6 t^6}{6!} + \frac{\omega^6 t^4}{4!(3\alpha\omega)^2} - \frac{\omega^6 t^2}{2!(3\alpha\omega)^4} + \frac{\omega^6}{(3\alpha\omega)^6} - \frac{\omega^6}{(3\alpha\omega)^6}\cos 3\alpha\omega t\right], \tag{40}$$

$$\vdots$$

In this procedure, four components of the decomposition series are obtained.

Continuing the expansion to the last term, series converges to the solution as follows:

$$\begin{aligned}
v_1(t) &= v_{01}(t) + v_{11}(t) + v_{21}(t) + v_{31}(t) + \cdots \\
&= \frac{\varepsilon A^3}{36}\left[\left(1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} - \frac{\omega^6 t^6}{6!} + \cdots\right) + \frac{\omega^2}{(3\alpha\omega)^2}\left(1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} - \frac{\omega^6 t^6}{6!} + \cdots\right)\right. \\
&+ \frac{\omega^2}{(3\alpha\omega)^2}\left(1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} - \frac{\omega^6 t^6}{6!} + \cdots\right) + \frac{\omega^4}{(3\alpha\omega)^4}\left(1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} - \frac{\omega^6 t^6}{6!} + \cdots\right) \\
&\left. - \cos 3\alpha\omega t\left(1 + \frac{\omega^2}{(3\alpha\omega)^2} + \frac{\omega^4}{(3\alpha\omega)^4} + \frac{\omega^6}{(3\alpha\omega)^6} + \cdots\right)\right] \\
&= \frac{\varepsilon A^3}{36}\left[\cos\omega t + \cos\omega t\left(\frac{\omega^2}{(3\alpha\omega)^2} + \frac{\omega^4}{(3\alpha\omega)^4} + \cdots\right) - \cos 3\alpha\omega t \cdot \frac{9\alpha^2\omega^2}{9\alpha^2\omega^2 - \omega^2}\right] \\
&= \frac{\varepsilon A^3}{36} \cdot \frac{9\alpha^2\omega^2}{9\alpha^2\omega^2 - \omega^2}(\cos\omega t - \cos 3\alpha\omega t).
\end{aligned} \tag{41}$$

$$v_1(t) = -\frac{\alpha^2\omega^2\varepsilon A^3}{4(9\alpha^2-1)}(\cos 3\alpha t - \cos t). \tag{42}$$

If the first-order approximation be sufficient, we have:

$$u_1(t) = \lim_{p\to 1} v_1(t) = v_0(t) + v_1(t) = A\cos\alpha\omega t - \frac{\alpha^2\omega^2\varepsilon A^3}{4(9\alpha^2-1)}(\cos 3\alpha t - \cos t). \tag{43}$$

Which is exactly the same as those obtained by variational iteration method [8].

3. Conclusion

In this paper, it does seem to be ture that the researcher has successfully used HPM and ADM to nonlinear equations. Besides, in HPM, we should solve a set of differential equations. Moreover, ADM has been employed for solving equations which are obtained from HPM. In the given examples, the researcher has obtained first- order approximation of solutions. Examples show that the results of the ADM method are in excellent agreement with those obtained by the variational method [8]. It has been apparently proved that these methods are powerful and efficient for solving nonlinear equation.

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Department of Mathematics, Payame Noor University, Tehran, Iran
Email address: skhaleghizadeh@yahoo.com,

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