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On the zeros and critical points of a polynomial

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Abstract. Let $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ be a polynomial of degree n. The Gauss-Lucas Theorem asserts that the zeros of the derivative $P'(z) = a_1 + \cdots + (n-1)a_{n-1}z^{n-2} + nz^{n-1}$, lie in the convex hull of the zeros of $P(z)$. Given a zero of $P(z)$ or $P'(z)$, A. Aziz [1], determined regions which contain at least one zero of $P(z)$ or $P'(z)$ respectively. In this paper, we give simple proofs and improved version of various results proved in [1], concerning the zeros of a polynomial and its derivative.

1. Introduction

Let a polynomial $P(z)$ of degree n has all it's zeros in $|z| \leq 1$. The Gauss-Lucas Theorem [4], asserts that all its critical points also lie in $|z| \leq 1$. Let $P(z^*) = 0$, then the famous Sendov's conjecture [4], says that the closed disk $|z - z^*| \leq 1$ contains a critical point of $P(z)$, (i.e. a zero of $P'(z)$). The conjecture has been proved for the polynomials of degree at most eight [2]. Also, the conjecture is true for some special class of polynomials such as the polynomials having a zero at the origin and the polynomials having all their zeros on $|z| = 1$, as shown in [2]. However, the general version is still unproved. Aziz[1], proved the following results regarding the relationship between the zeros and critical points of a polynomial.

Theorem 1.1. *If $P(z)$ is a polynomial of degree n and ω is a zero of $P'(z)$, then for every given real or complex number α, $P(z)$ has at least one zero in the region*

$$\left|\omega - \frac{\alpha + z}{2}\right| \leq \left|\frac{\alpha - z}{2}\right|. \tag{1}$$

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Theorem 1.2. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \leq 1$ and $P(a) = 0, a \neq 0$, then for every positive integer m, the polynomial $F(z) = mP(z) + zP'(z)$ has at least one zero in the disk*

$$|z - a| \leq 1. \tag{2}$$

Theorem 1.3. *If $P(z) = (z-a)Q(z)$ is a polynomial of degree n and if all the zeros of $Q(z)$ lie in the disk $|z + \alpha - a| \leq |\alpha|$ for some real or complex number $\alpha \neq 0$, then at least one zero of $P'(z)$ lies in the disk*

$$\left|z - a + \frac{\alpha}{2}\right| \leq \left|\frac{\alpha}{2}\right|. \tag{3}$$

2. Main Results

In this paper, we give simple proofs and improved version of various results proved in [1], concerning the zeros of a polynomial and its derivative.In the first result, we not only give a simple proof, but also an improved version of Theorem 1.1.

Theorem 2.1. *Let $P(z)$ be a polynomial of degree n and $P'(w) = 0$, then for every given real or complex number α, $P(z)$ has at least one zero in each of the regions*

$$\left|w - \frac{\alpha + z}{2}\right| \leq \left|\frac{\alpha - z}{2}\right| \tag{4}$$

and

$$\left|w - \frac{\alpha + z}{2}\right| \geq \left|\frac{\alpha - z}{2}\right|. \tag{5}$$

By using Lemma 3.2, we will now prove that Sendov's Conjecture is true incase $P(0) = 0$.

Theorem 2.2. *Let $P(z)$ be a polynomial having all it's zeros in $|z| \leq 1$ and $P(0) =$ 0. Then, for any zero z^* of $P(z)$, the closed disk $|z - z^*| \leq 1$ contains a zero of $P'(z)$.*

The next result shows that Theorem 1.3, is a simple consequence of Lemma 3.3, by making a simple transformation.

Theorem 2.3. *If $P(z) = (z-a)Q(z)$ is a polynomial of degree n and if all the zeros of $Q(z)$ lie in the region $|z + \alpha - a| \leq |\alpha|$ for some real or complex number $\alpha \neq 0$, then atleast one zero of $P'(z)$ lie in the region*

$$\left|z - a + \frac{\alpha}{2}\right| \leq \frac{|\alpha|}{2}. \tag{6}$$

3. Lemmas

However, for the proof of our results, we need the following lemmas.

Lemma 3.1. *(Laguerre Theorem) If all the zeros of the polynomial $P(z)$ lie in the circular domain K and if w is any zero of polar derivative $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$, then not both α and w lie outside K.*

Lemma 3.2. *If $P(z)$ is a polynomial of degree n such that $P(z_1) = P(z_2), z_1 \neq z_2$, then $P'(z)$ has at least one zero in each of the regions*

$$|z - z_1| \leq |z - z_2|$$

and

$$|z - z_1| \geq |z - z_2|. \tag{7}$$

This result is a simple consequence of Grace's Theorem [4]. The following result is due to Goodman, Rahman and Ratti [3].

Lemma 3.3. *If $P(z)$ is a polynomial of degree n having all it's zeros in $|z| \leq 1$ and $P(1) = 0$, then the disk $|z - \frac{1}{2}| \leq \frac{1}{2}$ contains a zero of $P'(z)$.*

4. Proofs

Proof of Theorem 2.1: We observe that the regions

$$\left|w - \frac{\alpha + z}{2}\right| \leq \left|\frac{\alpha - z}{2}\right|$$

and

$$\left|w - \frac{\alpha + z}{2}\right| \geq \left|\frac{\alpha - z}{2}\right|$$

are respectively equivalent to the regions

$$|z - (2w - \alpha)| \leq |z - \alpha|$$

and

$$|z - (2w - \alpha)| \geq |z - \alpha|.$$

These two regions are simply the right and left half planes respectively, formed by a line passing through w. The direction of the line depends on the point α. Since w is a critical point of $P(z)$, therefore, by Gauss - Lucas Theorem, there are zeros of $P(z)$ in both the half planes . This proves the theorem.
Proof of Theorem 2.2. By Gauss- Lucas Theorem, all zeros of $P'(z)$ lie in $|z| \leq 1$. Also,

$$P(0) = 0 = P(z^*).$$

By Lemma 3.2, the region $|z - z^*| \leq |z - 0| = |z|$ contains a zero w of $P'(z)$. But $|w| \leq 1$, and hence

$$w \in \{z \in C / |z - z^*| \leq |z|\} \cap \{z \in C / |z| \leq 1\}.$$

From the above, we observe that w satisfies the inequality $|w - z^*| \leq 1$. This proves the theorem.

Proof of Theorem 2.3. For any $\alpha \neq 0 \in C$, we consider the polynomial

$$G(z) = P\left(\alpha z + (a - \alpha)\right).$$

Then, $G(z)$ has all its zero in $|z| \leq 1$ and $G(1) = P(a) = 0$. Thus, by Lemma 3.3, the disk $\left|z - \frac{1}{2}\right| \leq \frac{1}{2}$ contains a zero of

$$G'(z) = \alpha P'\left(\alpha z + (a - \alpha)\right).$$

So, let $G'(z^*) = 0$, with $\left|z^* - \dfrac{1}{2}\right| \leq \dfrac{1}{2}$, then $w = \alpha z^* + a - \alpha$, is a zero of $P'(z)$, which is contained in the disk $\left|z - a + \dfrac{\alpha}{2}\right| \leq \dfrac{|\alpha|}{2}$. This proves the result.

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