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# Perturbed second-order state-dependent Moreau's sweeping process

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Abstract. In this paper, using a discretization approach, the existence of solutions for a class of second-order differential inclusions is stated in finite dimensional setting. The right hand side of the problem is governed by the so-called nonconvex state-dependent sweeping process and contains a general perturbation with unbounded values.

## 1. Introduction

The sweeping process is a very useful and wonderful model in plasticity and friction dynamics. Introduced by J. J. Moreau, see for instance [27], and motivated by its applications in unilateral mechanics for modeling the quasi-static evolution of elastoplastic systems, the sweeping process has found applications in nonsmooth mechanics, mathematical economics and planning procedures, electrical circuits, crowd motion modeling and other fields, see the recent paper [31] and the references therein. The problem lies in finding a trajectory $u(t) \in K(t)$, $t \in [T_0, T]$ satisfying the following Cauchy problem

$$\begin{cases} -\dot{u}(t) \in N_{K(t)}(u(t)), \quad \text{a.e. } t \in [T_0, T]; \\ u(t) \in K(t), \ \forall t \in [T_0, T], \end{cases}$$

where $N_{K(t)}(u(t))$ is the (outward) normal cone to the moving convex and closed set $K(t)$ at the point $u(t)$. Let recall the geometrical-mechanical interpretation of the sweeping process from [27]: as long as the point $u(t)$ happens to be in the interior of $K(t)$, the normal cone $N_{K(t)}(u(t))$ is reduced to zero, so $u(t)$ does not move. When

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the point is caught up by the boundary of $K(t)$, it moves, subject to an inward normal direction, as if it is swept by the moving set $K(t)$.

This problem has been the subject of various extensions in several directions, including generalizations to the non-convex case, the perturbed problem, the state-dependent case, as well as the second-order problem, see for instance [**2, 3, 4, 5, 9, 11, 12, 13, 15, 21, 32**]. Recently, the second-order perturbed state-dependent non-convex sweeping process has been a particular attraction for many authors, it takes the following form: let $H$ be a Hilbert space, $T_0$ and $T$ be two non-negative real numbers with $0 \leq T_0 < T$, and let for each $t \in [T_0, T]$ and $x \in H$, a nonempty closed subset $K(t, x)$ of $H$. Given $b \in H$ and $a \in K(T_0, b)$, we have to find two absolutely continuous mappings $u, v : [T_0, T] \to H$ with $u(t) \in K(t, v(t))$ for all $t \in [T_0, T]$ satisfying

$$(P_F) \begin{cases} -\dot{u}(t) \in N_{K(t,v(t))}(u(t)) + F(t, v(t), u(t)), \quad \text{a.e. } t \in [T_0, T]; \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds, \quad u(t) = a + \int_{T_0}^{t} \dot{u}(s)ds, \ \forall t \in [T_0, T]; \\ \qquad u(t) \in K(t, v(t)), \ \forall t \in [T_0, T], \end{cases}$$

where $N_{K(t,v(t))}(u(t))$ denotes the normal cone to $K(t, v(t))$ at the point $u(t)$, $F : [T_0, T] \times H \times H \to H$ is a single or set-valued mapping. The differential inclusion $(P_F)$ was studied for the first time when the sets $K(t, v(t))$ are convex and compact and $F \equiv 0$ by C. Castaing [**14**], then by K. Chraibi [**18**] and Kunze and Monteiro-Marques [**25**]. The non-convex case has been considered by Chemetov and Monteiro-Marques [**17**], they proved the existence of solutions to $(P_F)$ for uniformly prox-regular sets $K(t, v(t)))$ with absolutely continuous variation in space, Lipschitz variation in time and with a single-valued perturbation. By means of a generalized version of the Shauder theorem, Castaing, Ibrahim and Yarou [**16**] provided an other approach to prove the existence for uniformly prox regular and ball-compact sets $K(t, v(t))$ with absolutely continuous variation in time, without perturbation and for the perturbed problem (even in presence of a delay). The existence of solution for such problem is established by proving the convergence of the Moreau's catching-up algorithm. Later, other approaches were used: one consists, in the finite dimensional setting, in reducing the constrained differential inclusion $(P_F)$ to an unconstrained one governed by the subdifferential of the distance function, see [**22**]; the second consists in approaching the problem by a regularized one depending on a positif parameter converging to zero, even for a general class of sets, namely equi-uniformly subsmooth sets and positively $\alpha$-far sets, see [**7, 8, 23**]. For recent other results, we refer to [**1, 6, 30, 33**].

A classic approach to solving second order problems consists of first order reduction and use of known results. Generally, this is made possible thanks to the use of fixed point theory. This was obtained in a recent paper [**24**] with strong conditions: the sets $K(t, x)$ are contained in a strong compact, moreover only the particular case of a single-valued perturbation satisfying the linear growth condition has been considered. Very recently, [28] and [34] have presented a new approach for solving second-order sweeping process with set-valued perturbation in the finite dimensional setting: it consists in a reduction of the problem to a first order perturbed sweeping process and a use of the known results in this case without use of fixed point theory nor any compactness condition.

Our main purpose in this paper is to study, in the finite dimensional setting, the second-order sweeping process with two perturbations ($F = G + g$)

$$(\mathcal{P}) \begin{cases} -\dot{u}(t) \in N_{K(t,v(t))}(u(t)) + G(t, v(t), u(t)) + g(t, v(t), u(t)) \text{ a.e } t \in [T_0, T] \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds, \quad u(t) = a + \displaystyle\int_{T_0}^{t} \dot{u}(s)ds, \ \forall t \in [T_0, T], \\ u(t) \in K(t, v(t)), \ \forall t \in [T_0, T], \end{cases}$$

where $G : [T_0, T] \times H \times H \rightharpoonup H$ is an upper semicontinuous set-valued map with nonempty closed convex values unnecessarily bounded and without any compactness condition and $g : [T_0, T] \times H \times H \to H$ is a single-valued mapping satisfying the linear growth condition. Our aim in this paper is twofold: using the recent result of [28] and taking a perturbation as a sum of two mappings with single and set-values respectively, we generalize all the results obtained in the two cases. Using a different approach, we weaken the hypotheses on the perturbation by taking a Carathéodory mapping satisfying a linear growth condition and an unbounded set-valued perturbation for which only the element of minimum norm satisfies a linear growth condition.

The paper is organized as follows. In Section 2, we recall some basic notation, definitions and useful results which are used throughout the paper. In Section 3, we state the existence results.

## 2. Notation and Preliminaries

Let $H$ be a real Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$, and the associated norm by $\| \cdot \|$. We denote by $\overline{\mathbf{B}}_H$ the unit closed ball of $H$, $L^1_H([T_0, T])$ the space of all Lebesgue-Bochner integrable $H$-valued mappings defined on $[T_0, T]$, by $\mathcal{C}_H([T_0, T])$ the Banach space of all continuous mappings $u : [T_0, T] \to H$ endowed with the norm of uniform convergence and $AC([T_0, T])$ the space of absolutely continuous mapping.

For any nonempty subsets $S, S'$ of $H$, we denote by:

• $d(\cdot, S)$ the usual distance function associated with $S$, and $\delta^*(x'; S) = \sup_{y \in S} \langle x', y \rangle$ the support function of $S$ at $x' \in H$. If $S$ is closed convex, we have

$$d(x, S) = \sup_{x' \in \overline{\mathbf{B}}_H} [\langle x', x \rangle - \delta^*(x'; S)].$$

• $Proj_S(u)$ the projection of $u$ onto $S$, defined by

$$Proj_S(u) = \{y \in S : \quad d(u,S) = \|u-y\|\},$$

it is unique whenever $S$ is closed convex.
• $\mathcal{H}$ the Hausdorff distance between $S$ and $S'$, defined by

$$\mathcal{H}(S,S') = \max\{\sup_{u\in S} d(u,S'), \sup_{v\in S'} d(v,S)\}.$$

• $co(S)$ the convex hull of $S$ and $\overline{co}(S)$ its closed convex hull, characterized by

$$\overline{co}(S) = \{x \in H : \ \forall x' \in H, \ \langle x', x\rangle \le \delta^*(x';S)\}.$$

We need in the sequel to recall some definitions and results that will be used throughout the paper. Let $A$ be an open subset of $H$ and $\varphi : A \to (-\infty,+\infty]$ be a lower semicontinuous function, the proximal subdifferential $\partial^P\varphi(x)$, of $\varphi$ at $x$ (see [**20**]) is the set of all proximal subgradients of $\varphi$ at $x$. Any $\xi \in H$ is a proximal subgradient of $\varphi$ at $x$ if there exist positive numbers $\eta$ and $\varsigma$ such that

$$\varphi(y) - \varphi(x) + \eta\|y-x\|^2 \ge \langle \xi, y-x\rangle, \ \forall y \in x + \varsigma\overline{\mathbf{B}}_H.$$

Let $x$ be a point of $S \subset H$, we recall (see [**20**]) that the proximal normal cone to $S$ at $x$ is defined by $N^P(S,x) = \partial^P\Psi_S(x)$, where $\Psi_S$ denotes the indicator function of $S$, i.e. $\Psi_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N_S^P(x) = \{\xi \in H : \ \exists \varrho > 0 \text{ s.t } x \in \mathrm{Proj}_S(x+\varrho\xi)\}.$$

If $\varphi$ is a real-valued locally-Lipschitz function defined on $H$, the Clarke subdifferential $\partial^C\varphi(x)$ of $\varphi$ at $x$ is the nonempty convex compact subset of $H$ given by

$$\partial^C\varphi(x) = \{\xi \in H : \ \varphi^\circ(x;v) \ge \langle \xi, v\rangle, \forall v \in H\},$$

where

$$\varphi^\circ(x;v) = \lim_{y\to x,}\sup_{t\downarrow 0} \frac{\varphi(y+tv)-\varphi(y)}{t}$$

is the generalized directional derivative of $\varphi$ at $x$ in the direction $v$ (see [**20**]). The Clarke normal cone $N^C(S,x)$ to $S$ at $x \in S$ is defined by polarity with $T_S^C$, that is,

$$N_S^C(x) = \{\xi \in H : \ \langle \xi, v\rangle \le 0, \ \forall v \in T_S^C\},$$

where $T_S^C$ denotes the Clarke tangent cone and is given by

$$T_S^C = \{v \in H : \ d^\circ(x,S;v) = 0\}.$$

Recall now, that for a given $\rho \in ]0,+\infty]$ the subset $S$ is uniformly $\rho$-prox regular (see [29]) or equivalently $\rho$-proximally if every nonzero proximal to $S$ can be realized by a $\rho$-ball, this means that for all $\overline{x} \in S$ and all $0 \neq \xi \in N_S^P(\overline{x})$, one has

$$\langle \frac{\xi}{\|\xi\|}, x - \overline{x} \rangle \leq \frac{1}{2\rho} \|x - \overline{x}\|^2,$$

for all $x \in S$. We make the convention $\frac{1}{\rho} = 0$ for $\rho = +\infty$. Recall that for $\rho = +\infty$ the uniform $\rho$-prox regularity of $S$ is equivalent to the convexity of $S$. It's well known that the class of uniformly $\rho$-prox regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of a Hilbert space and many other non-convex sets (see [19, 29]).

The following proposition summarizes some important properties of the uniform prox-regularity needed in the sequel. For the proof of these results we refer the reader to [12] and [29].

**Proposition 2.1.** *Let $S$ be a nonempty closed subset of $H$ and $x \in S$. The following assertions hold:*
***1)*** $\partial^P d(x,S) = N_S^P(x) \cap \overline{\mathbf{B}}_H$;
***2)*** *if $S$ is uniformly $\rho$-prox-regular with $\rho \in ]0,+\infty]$, then*

***i)*** *the proximal subdifferential of $d(\cdot,S)$ coincides with its Clarke subdifferential at all points $x \in H$ satisfying $d(x,S) < \rho$. So, $\partial d(x,S) = \partial^P d(x,S) = \partial^C d(x,S)$ is closed convex;*

***ii)*** *the proximal normal cone to $S$ coincides with all normal cones contained in the Clarke normal cone at all points $x \in S$, i.e., $N_S(x) = N_S^P(x) = N_S^C(x)$;*

***iii)*** *let $K : [T_0,T] \times H \rightharpoonup H$ be a uniformly $\rho$-prox regular closed valued mapping satisfying*

$$\left| d\big(x_1, K(t,y_1)\big) - d\big(x_2, K(s,y_2)\big) \right| \leq \|x_1 - x_2\| + \zeta(t) - \zeta(s) + L\|y_1 - y_2\|;$$

*for all $s \leq t$ in $[T_0,T]$ and $x_i, y_i$ in $H\,(i = 1,2)$, where $\zeta : [T_0,T] \to \mathbf{R}^+$ is a nondecreasing absolutely continuous function and $L$ is a positive constant. Then the convex weakly compact valued mapping $(t,x,y) \to \partial^p d(y,K(t,x))$ satisfies the upper semicontinuity property.*

## 3. Main result

Let $H = \mathbf{R}^n$ the finite dimensional Euclidean space. The following assumptions will be useful.

**Assumption 1.** Let $K : [T_0,T] \times \mathbf{R}^n \rightharpoonup \mathbf{R}^n$ be a set-valued mapping with nonempty closed values, such that:

($\mathcal{K}_1$) there is a positive constant $0 < L < 1$ and a nondecreasing absolutely continuous function $\zeta : [T_0,T] \to \mathbf{R}^+$ such that, for all $s \leq t$ in $[T_0,T]$ and

$x_i, y_i$ in $\mathbf{R}^n (i = 1, 2)$,

$$\left|d\big(x_1, K(t, y_1)\big) - d\big(x_2, K(s, y_2)\big)\right| \leq \|x_1 - x_2\| + \zeta(t) - \zeta(s) + L\|y_1 - y_2\|;$$

($\mathcal{K}_2$) there exists some constant $\rho \in ]0, +\infty]$ such that for each $(t, y) \in [T_0, T] \times \mathbf{R}^n$ the sets $K(t, y)$ are uniformly $\rho$-prox regulars.

**Assumption 2.** Let $g : [T_0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be a mapping satisfying:

($\mathcal{H}_1$) for any fixed $(x, y) \in \mathbf{R}^n$, $g(\cdot, x, y)$ is Lebesgue measurable on $[T_0, T]$;
($\mathcal{H}_2$) for any fixed $t \in [T_0, T]$, $g(t, \cdot, \cdot)$ is continuous on $\mathbf{R}^n \times \mathbf{R}^n$;
($\mathcal{H}_3$) there exists a non-negative function $p(\cdot) \in \mathbf{L}^1_{\mathbf{R}^+}([T_0, T])$ such that, for all $(t, x, y) \in [T_0, T] \times \mathbf{R}^n \times \mathbf{R}^n$,

$$\|g(t, x, y)\| \leq p(t)(1 + \|x\| + \|y\|).$$

**Assumption 3.** Let $G : [T_0, T] \times \mathbf{R}^n \times \mathbf{R}^n \rightharpoonup \mathbf{R}^n$ be a set-valued mapping with nonempty closed convex values satisfying:

($\mathcal{G}_1$) $G(\cdot, \cdot, \cdot)$ is upper semicontinuous on $[T_0, T] \times \mathbf{R}^n \times \mathbf{R}^n$;
($\mathcal{G}_2$) there exists a real $q > 0$, such that, for all $(t, x, y) \in [T_0, T] \times \mathbf{R}^n \times \mathbf{R}^n$,

$$d(0, G(t, x, y)) \leq q(1 + \|x\| + \|y\|).$$

Let us start with an existence result for second-order state-dependent sweeping process without perturbation. It will be used in the next proposition. The proof is a careful adaptation of Theorem 3.1 in [**28**].

**Theorem 3.1.** *Assume that Assumption 1 holds. Then, for every $b \in \mathbf{R}^n$ and for every $a \in K(T_0, b)$, there exists $(u, v) \in AC([T_0, T]) \times AC([T_0, T])$ such that*

$$(\mathcal{I}) \begin{cases} -\dot{u}(t) \in N_{K(t, v(t))}(u(t)), \ a.e. \ t \in [T_0, T], \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds, \ \ u(t) = a + \displaystyle\int_{T_0}^{t} \dot{u}(s)ds, \ \forall t \in [T_0, T], \\ u(t) \in K(t, v(t)), \ \forall t \in [T_0, T], \end{cases}$$

*with*

$$\|(\dot{u}(t), \dot{v}(t))\| \leq \frac{1 + \dot{\zeta}(t)}{1 - L} \quad a.e. \quad t \in [T_0, T].$$

We can easily deduce the following result for second-order differential inclusion with a single-valued perturbation $h \in L^1_{\mathbf{R}^n}([T_0, T])$.

**Proposition 3.2.** *Assume that Assumption 1 holds. Then, for any mapping $h \in L^1_{\mathbf{R}^n}([T_0, T])$ and for every $b \in \mathbf{R}^n$ and every $a \in K(T_0, b)$, there exists $(u, v) \in AC([T_0, T]) \times AC([T_0, T])$ satisfying*

$$(\mathcal{II}) \begin{cases} -\dot{u}(t) \in N_{K(t, v(t))}(u(t)) + h(t), \ a.e. \quad t \in [T_0, T], \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds, \ \ u(t) = a + \displaystyle\int_{T_0}^{t} \dot{u}(s)ds, \ \forall t \in [T_0, T], \\ u(t) \in K(t, v(t)), \ \forall t \in [T_0, T]. \end{cases}$$

*Moreover, we have*

$$\|(\dot{u}(t),\dot{v}(t))\| \leq \frac{1}{1-L}\left(1+\dot{\zeta}(t)+2\Big(\|h(t)\|+\int_{T_0}^{t}\|h(\tau)\|d\tau\Big)\right).$$

PROOF. For any $t\in[T_0,T]$, we put $\varphi(t)=\int_{T_0}^{t}h(s)ds$ and $\psi(t)=\int_{T_0}^{t}\varphi(s)ds$. Consider the set-valued mapping $C:[T_0,T]\times\mathbf{R}^n\rightharpoonup\mathbf{R}^n$ defined by

$$C(t,y)=K(t,y-\psi(t))+\varphi(t),\ \ \forall\ (t,y)\in[T_0,T]\times\mathbf{R}^n.$$

Obviously, $C$ satisfies $(\mathcal{K}_2)$, let verify $(\mathcal{K}_1)$. For any $x_i,y_i$ in $\mathbf{R}^n(i=1,2)$ and any $s\leq t$ in $[T_0,T]$, we have

$$\begin{aligned}
&|d(x_1,C(t,y_1))-d(x_2,C(s,y_2))|\\
&=|d(x_1-\varphi(t),K(t,y_1-\psi(t)))-d(x_2-\varphi(s),K(s,y_2-\psi(s)))|\\
&\leq\|x_1-x_2\|+\|\varphi(t)-\varphi(s)\|+\zeta(t)-\zeta(s)+L\big(\|\psi(t)-\psi(s)\|+\|y_1-y_2\|\big)\\
&\leq\|x_1-x_2\|+\zeta_1(t)-\zeta_1(s)+L\|y_1-y_2\|
\end{aligned}$$

where

$$\zeta_1(t)=\int_{T_0}^{t}\left(\dot{\zeta}(\tau)+\|h(\tau)\|+L\|\varphi(\tau)\|\right)d\tau$$

is nondecreasing absolutely continuous. Hence, $C$ satisfies $(\mathcal{K}_1)$, as $a\in C(T_0,b)=K(T_0,b)$, from Theorem 3.1, there exists $(w,z)\in AC([T_0,T])\times AC([T_0,T])$ such that

$$\begin{cases}
-\dot{z}(t)\in N_{C(t,w(t))}(z(t))\ \text{ a.e. }\ t\in[T_0,T];\\
w(t)=b+\displaystyle\int_{T_0}^{t}w(s)ds,\ \ z(t)=a+\int_{T_0}^{t}\dot{z}(s)ds,\ \forall t\in[T_0,T];\\
z(t)\in C(t,w(t)),\ \forall t\in[T_0,T],
\end{cases}$$

with $\|(\dot{z}(t),\dot{w}(t))\|\leq\dfrac{1+\dot{\zeta}_1(t)}{1-L}$ a.e. $t\in[T_0,T]$. Let defined $(u(\cdot),v(\cdot))$ by $(u(t),v(t))=(z(t)-\varphi(t),w(t)-\psi(t)$, then, for a.e. $t\in[T_0,T]$,

$$\|(\dot{u}(t)+h(t),\dot{v}(t)+\varphi(t))\|\leq\frac{1+\dot{\zeta}_1(t)}{1-L}$$

so that, we have

$$\|\dot{u}(t)\|\leq\frac{1+\dot{\zeta}(t)+\|h(t)\|+L\|\varphi(t)\|}{1-L}+\|h(t)\|$$

and

$$\|\dot{v}(t)\|\leq\frac{1+\dot{\zeta}(t)+\|h(t)\|+L\|\varphi(t)\|}{1-L}+\|\varphi(t))\|$$

since $0<L<1$, we can write

$$\|\dot{u}(t)\|\leq\frac{1+\dot{\zeta}(t)+2\|h(t)\|+\|\varphi(t)\|}{1-L}$$

and

$$\|\dot{v}(t)\| \leq \frac{1+\dot{\zeta}(t)+\|h(t)\|+2\|\varphi(t)\|}{1-L}.$$

So, we conclude that

$$\|(\dot{u}(t),\dot{v}(t))\| \leq \frac{1+\dot{\zeta}(t)+2\Big(\|h(t)\|+\|\varphi(t)\|\Big)}{1-L}.$$

Consequently $(u(\cdot),v(\cdot))$ satisfies $(\mathcal{II})$, and we have

$$\|(\dot{u}(t),\dot{v}(t))\| \leq \frac{1}{1-L}\Bigg(1+\dot{\zeta}(t)+2\Big(\|h(t)\|+\int_{T_0}^{t}\|h(\tau)\|d\tau\Big)\Bigg).$$

The proof is then complete. □

Now, we give the main result.

**Theorem 3.3.** *Assume that Assumptions 1, 2 and 3 hold. Then, for every $b \in \mathbf{R}^n$ and every $a \in K(T_0,b)$, there exists $(u,v) \in AC([T_0,T]) \times AC([T_0,T])$ satisfying $(\mathcal{P})$.*

Proof. For each $(t,x,y) \in [T_0,T]\times\mathbf{R}^n\times\mathbf{R}^n$, denote by $\pi(\cdot,\cdot,\cdot)$ the element of minimal norm of the closed convex set $G(t,x,y)$ of $\mathbf{R}^n$, that is

$$\pi(t,x,y) = \mathrm{Proj}_{G(t,x,y)}(0).$$

We put $f(t,(x,y)) = g(t,x,y)+\pi(t,x,y)$ and $\beta(t)=p(t)+q$, by $(\mathcal{H}_1),(\mathcal{H}_3)$ and $(\mathcal{G}_1)$, $f(\cdot,(x,y))$ is Lebesgue measurable and for all $(t,x,y) \in [T_0,T]\times\mathbf{R}^n\times\mathbf{R}^n$,

$$\|f(t,(x,y))\| \leq \beta(t)(1+\|(x,y)\|). \tag{1}$$

We suppose that

$$\int_{T_0}^{T}\beta(s)ds \leq \frac{1-L}{4(1+T)}. \tag{2}$$

**Construction of sequences.** Consider, for every $n \in \mathbf{N}$, a partition of $[T_0,T]$ defined by $t_i^n = T_0 + i\frac{T-T_0}{n}$ $(0 \leq i \leq n)$. For $a \in K(T_0,b)$, let us consider the following problem on the interval $[T_0,t_1^n]$ :

$$(P_0)\begin{cases} -\dot{u}(t) \in N_{K(t,v(t))}(u(t)) + f\big(t,(b,a)\big) \text{ a.e. } t \in [T_0,t_1^n],\\ v(T_0)=b,\ u(T_0)=a \in K(T_0,b).\end{cases}$$

By Proposition 3.2, there exists an $AC([T_0,t_1^n]) \times AC([T_0,t_1^n])$ solution of $(P_0)$ that we denote by $(u_0^n,v_0^n)$. Now, since $u_0^n(t_1^n) \in K(t_1^n,v_0^n(t_1^n))$ is well defined, on the interval $[t_1^n,t_2^n]$ the problem

$$(P_1)\begin{cases} -\dot{u}_1^n(t) \in N_{K(t,v_1^n(t))}(u_1^n(t)) + f\big(t,v_0^n(t_1^n),(u_0^n(t_1^n))\big) \text{ a.e. } t \in [t_1^n,t_2^n],\\ u_0^n(t_1^n) \in K(t_1^n,v_0^n(t_1^n)),\end{cases}$$

admits an $AC([t_1^n,t_2^n])\times AC([t_1^n,t_2^n])$ solution $(u_1^n,v_1^n)$ with $(u_1^n(t_1^n),v_1^n(t_1^n))=(u_0^n(t_1^n),v_0^n(t_1^n))$. So, for each $n$, there exists a finite sequence $(u_i^n,v_i^n)\in AC([t_i^n,t_{i+1}^n])\times AC([t_i^n,t_{i+1}^n])$ with $(u_i^n(t_i^n),v_i^n(t_i^n))=(u_{i-1}^n(t_i^n),v_{i-1}^n(t_i^n))$ such that, for each $i\in\{0,...,n-1\}$,

$$(P_i)\begin{cases} -\dot u_i^n(t)\in N_{K(t,v_i^n(t))}(u_i^n(t))+f\big(t,(v_{i-1}^n(t_i^n),u_{i-1}^n(t_i^n))\big) \text{ a.e. } t\in[t_i^n,t_{i+1}^n],\\ u_{i-1}^n(t_i^n)\in K(t_i^n,v_{i-1}^n(t_i^n)),\end{cases}$$

where $(u_{-1}^n(T_0),v_{-1}^n(T_0))=(a,b)$ and a.e. $t\in[t_i^n,t_{i+1}^n]$

$$\|(\dot u_i^n(t),\dot v_i^n(t))\|\le\frac{1}{1-L}\Bigg(1+\dot\zeta(t)+2\|f\big(t,(v_{i-1}^n(t_i^n),u_{i-1}^n(t_i^n))\big)\|$$
$$+2\int_{t_i^n}^t\|f\big(\tau,(v_{i-1}^n(t_i^n),u_{i-1}^n(t_i^n))\big)\|d\tau\Bigg).$$

Define the mapping $(u_n,v_n):[T_0,T]\to\mathbf{R}^n\times\mathbf{R}^n$ by $(u_n(t),v_n(t))=(u_i^n(t),v_i^n(t))$, for all $t\in[t_i^n,t_{i+1}^n],\ i\in\{0,...,n-1\}$ and put

$$\theta_n(t)=\begin{cases} t_k^n & \text{if} \quad t\in[t_k^n,t_{k+1}^n[\\ T & \text{if} \quad t=t_n^n.\end{cases}$$

For all $n\in\mathbf{N},\ (u_n,v_n)\in AC([T_0,T])\times AC([T_0,T])$ and

$$\begin{cases} \dot u_n(t)\in -N_{K(t,v_n(t))}(u_n(t))+f\big(t,(v_n(\theta_n(t)),u_n(\theta_n(t)))\big) \ \ a.e.\ t\in[T_0,T],\\ u_n(t)\in K(t,v_n(t)),\ \forall t\in[T_0,T],\ u_n(T_0)=a,\ v_n(T_0)=b,\end{cases}$$

with a.e. $t\in[T_0,T]$

$$\|(\dot u_n(t),\dot v_n(t))\|\le\frac{1}{1-L}\Big(1+\dot\zeta(t)+2\|f\big(t,(v_n(\theta_n(t)),u_n(\theta_n(t)))\big)\|$$
$$+2\int_{\theta_n(t)}^t\|f\big(\tau,(v_n(\theta_n(t)),u_n(\theta_n(t)))\big)\|d\tau\Big). \tag{3}$$

**Convergence of the sequences.** By (3.3), we have

$$\|(u_n(t_{i+1}^n),v_n(t_{i+1}^n))\|\le\|(u_n(t_i^n),v_n(t_i^n))\|+\frac{1}{1-L}\int_{t_i^n}^{t_{i+1}^n}\Big(1+\dot\zeta(s)+$$
$$2(\|f(s,(v_n(t_i^n),u_n(t_i^n)))\|+\int_{t_i^n}^s\|f(\tau,(v_n(t_i^n),u_n(t_i^n)))\|d\tau)\Big)ds.$$

By iteration, we obtain

$$\|(u_n(t_{i+1}^n),v_n(t_{i+1}^n))\|\le\|(a,b)\|+\frac{1}{1-L}\Bigg(t_{i+1}^n+\sum_{k=0}^i\int_{t_k^n}^{t_{k+1}^n}\Big(\dot\zeta(s)+$$
$$2\big(\|f\big(s,(v_n(t_k^n),u_n(t_k^n))\big)\|+\int_{t_k^n}^s\|f\big(\tau,(v_n(t_k^n),u_n(t_k^n))\big)\|d\tau\big)\Big)ds\Bigg)$$
$$\le\|(a,b)\|+\frac{1}{1-L}\Big(t_{i+1}^n+\int_{T_0}^{t_{i+1}^n}\dot\zeta(s)ds+$$

$$2\sum_{k=0}^{i}\int_{t_k^n}^{t_{k+1}^n}\Big(\|f\big(s,(v_n(t_k^n),u_n(t_k^n))\big)\| + \int_{t_k^n}^{s}\|f\big(\tau,(v_n(t_k^n),u_n(t_k^n))\big)\|d\tau\Big)ds\Bigg).$$

By (1), one has

$$\|(u_n(t_{i+1}^n),v_n(t_{i+1}^n))\| \le \|(a,b)\| + \frac{1}{1-L}\Big(t_{i+1}^n + \int_{T_0}^{t_{i+1}^n}\dot\zeta(s)ds\Big) +$$

$$\frac{2(1+t_{i+1}^n-T_0)}{1-L}\sum_{k=0}^{i}\Big(1+\|(u_n(t_k^n),v_n(t_k^n))\|\Big)\int_{t_k^n}^{t_{k+1}^n}\beta(s)ds,$$

so, we get

$$\|(u_n(t_{i+1}^n),v_n(t_{i+1}^n))\| \le \|(a,b)\| + \frac{1}{1-L}\Big(t_{i+1}^n + \int_{T_0}^{t_{i+1}^n}\dot\zeta(s)ds\Big)$$

$$+\frac{2(1+t_{i+1}^n)}{1-L}\Big(1+\max_{0\le k\le i}\|(u_n(t_k^n),v_n(t_k^n)\|\Big)\int_{T_0}^{t_{i+1}^n}\beta(s)ds,$$

for each $i=0,...,n-1$, thus

$$\max_{0\le k\le n}\|(u_n(t_k^n),v_n(t_k^n))\| \le \|(a,b)\| + \frac{1}{1-L}\Big(T + \int_{T_0}^{T}\dot\zeta(s)ds\Big)$$

$$+\frac{2(1+T)}{1-L}\Big(1+\max_{0\le k\le n}\|(u_n(t_k^n),v_n(t_k^n))\|\Big)\int_{T_0}^{T}\beta(s)ds.$$

Taking in account (2), we obtain

$$\max_{0\le k\le n}\|(u_n(t_k^n),v_n(t_k^n))\| \le \|(a,b)\| + \frac{1}{1-L}\Big(T + \int_{T_0}^{T}\dot\zeta(s)ds\Big)$$

$$+\frac{1}{2}+\frac{1}{2}\max_{0\le k\le n}\|(u_n(t_k^n),v_n(t_k^n))\|.$$

Then, for all $n\in\mathbf{N}$,

$$\|(u_n(\theta_n(t)),v_n(\theta_n(t)))\| \le 1+2\Big(\|(a,b)\| + \frac{1}{1-L}\big(T+\int_{T_0}^{T}\dot\zeta(s)ds\big)\Big) := m. \quad (4)$$

By (1) and (4) one has for any $n$ and almost all $t\in[T_0,T]$

$$\|f(t,(u_n(\theta_n(t)),v_n(\theta_n(t)))\| \le (1+m)\beta(t) := r(t). \quad (5)$$

According to (3.3) and (3.5), one has

$$\|(\dot u_n(t),\dot v_n(t))\| \le \frac{1}{1-L}\Big(1+\dot\zeta(t)+2(1+T)r(t)\Big) := m_1(t). \quad (6)$$

As $\theta_n(t)\to t$ and $m_1\in \mathrm{L}^1_{\mathbf{R}^+}(T_0,T)$ it follows from (6) that

$$\lim_{n\to\infty}||(u_n(\theta_n(t)),v_n(\theta_n(t)))-(u_n(t),v_n(t))||=0, \quad (7)$$

and $(u_n, v_n)$ converges in $\mathcal{C}_{\mathbf{R}^n\times\mathbf{R}^n}([T_0,T])$ to $(u,v)$, thanks to Theorem 0.3.4 in [10]. Furthermore,

$$\|g(t, v_n(\theta_n(t)), u_n(\theta_n(t)))\| \leq (1+m)p(t),$$

by the continuity of the mapping $g(t,\cdot,\cdot)$ we get

$$g(t, v_n(\cdot), u_n(\cdot)) \to g(t, v(\cdot), u(\cdot)),$$

and

$$\|g(t, v(t), u(t))\| \leq (1+m)p(t).$$

On the other hand, we have

$$||\pi(t, v_n(\theta_n(t)), u_n(\theta_n(t)))\| \leq (1+m)q$$

for all $n \geq n_0$ and for all $t \in [T_0, T]$, we put

$$(\pi(\cdot, v_n(\theta_n(\cdot)), u_n(\theta_n(\cdot)))) = (\rho_n(\cdot)),$$

so $(\rho_n(\cdot))$ is integrably bounded, taking a subsequence if necessary, we may conclude that $(\rho_n(\cdot))$ converges $\sigma(L^1_{\mathbf{R}^n}, L^\infty_{\mathbf{R}^n})$ to some mapping $\rho(\cdot) \in L^1_{\mathbf{R}^n}([T_0,T])$ with

$$\|\rho(t)\| \leq q(1+m).$$

Now, we proceed to prove that

$$-\dot{u}(t) \in N_{K(t,v(t))}(u(t)) + G(t, v(t), u(t)) + g(t, v(t), u(t)) \text{ a.e. } \in [T_0, T].$$

First, we check that $u(t) \in K(t, v(t))$, for all $t \in [T_0,T]$. Indeed, for every $t \in [T_0,T]$ and for every $n$, we have

$$\begin{aligned} d(u_n(t), K(t, v(t))) &\leq ||u_n(t) - u_n(\theta_n(t))|| + d(u_n(\theta_n(t)), K(t, v(t))) \\ &\leq ||u_n(t) - u_n(\theta_n(t))|| + \mathcal{H}(K(\theta_n(t), v_n(\theta_n(t))), K(t, v(t))) \\ &\leq ||u_n(t) - u_n(\theta_n(t))|| + |\zeta(t) - \zeta(\theta_n(t))| + L||v_n(\theta_n(t)) - v_n(t)||, \end{aligned}$$

passing to the limit when $n \to \infty$, in the preceding inequality, we get $u(t) \in K(t, v(t))$.

On the other hand, if we put $(f(\cdot, v_n(\theta_n(\cdot)), u_n(\theta_n(\cdot)))) = (l_n(\cdot))$, $(l_n)$ converges $\sigma(L^1_{\mathbf{R}^n}, L^\infty_{\mathbf{R}^n})$ to $l$ with $\|l(t)\| \leq r(t)$, and we have

$$\|\dot{u}_n(t) - l_n(t)\| \leq \|\dot{u}_n(t)\| + \|l_n(t)\| \leq \lambda(t),$$

with $\lambda(t) = m_1(t) + r(t)$, then $-\dot{u}_n(t) + l_n(t) \in \lambda(t)\overline{\mathbf{B}}_{\mathbf{R}^n}$, since $-\dot{u}_n(t) + l_n(t) \in N_{K(t,v_n(t))}(u_n(t))$, we get by (1) of Proposition 2.1

$$-\dot{u}_n(t) + l_n(t) \in +\lambda(t)\partial d(u_n(t), K(t, v_n(t))).$$

Remark that $(-\dot{u}_n + l_n, \rho_n)$ weakly converges in $L^1_{\mathbf{R}^n\times\mathbf{R}^n}([T_0,T])$ to $(-\dot{u} + l, \rho)$. An application of the Mazur's theorem to $(-\dot{u}_n + l_n, \rho_n)$ provides a sequence $(w_n, \zeta_n)$ with

$$w_n \in co\{-\dot{u}_m + l_m : \ m \geq n\} \quad \text{and} \quad \zeta_n \in co\{\rho_m : \ m \geq n\}$$

such that $(w_n, \zeta_n)$ converges strongly in $L^1_{\mathbf{R}^n\times\mathbf{R}^n}([T_0,T])$ to $(-\dot{u}+l,\rho)$. We can extract from $(w_n,\zeta_n)$ a subsequence which converges a.e. to $(-\dot{u}+l,\rho)$. Then, there is a Lebesgue negligible set $S \subset [T_0,T]$ such that for every $t \in [T_0,T]\setminus S$

$$-\dot{u}(t)+l(t) \in \bigcap_{n\geq 0}\overline{\{w_m(t):\ m\geq n\}} \subset \bigcap_{n\geq 0}\overline{co}\{-\dot{u}_m(t)+l_m(t):\ m\geq n\} \tag{3.8}$$

$$\rho(t) \in \bigcap_{n\geq 0}\overline{\{\zeta_m(t):\ m\geq n\}} \subset \bigcap_{n\geq 0}\overline{co}\{\rho_m(t):\ m\geq n\}. \tag{3.9}$$

Fix any $t\in[T_0,T]\setminus S,\ n\geq n_0$ and $\mu\in\mathbf{R}^n$, then the relation (3.8) gives

$$\begin{aligned}\langle \mu, -\dot{u}(t)+l(t)\rangle &\leq \limsup_{n\to\infty}\delta^*(\mu,\lambda(t)\partial d(u_n(t),K(t,v(t))) \\ &\leq \delta^*(\mu,\lambda(t)\partial d(u(t),K(t,v(t))),\end{aligned}$$

where the first inequality follows from the characterization of convex hull and the second one follows from Proposition 2.1. Taking the supremum over $\mu\in\mathbf{R}^n$, we deduce that

$$\delta(-\dot{u}(t)+l(t),\lambda(t)\partial d(u(t),K(t,v(t))) =$$

$$\delta^{**}(-\dot{u}(t)+l(t),\lambda(t)\partial d(u(t),K(t,v(t))) \leq 0,$$

which entails

$$-\dot{u}(t)+l(t)\in\lambda(t)\partial d(u(t),K(t,v(t)))\subset N_{K(t,v(t))}(u(t)).$$

Further, the relation (3.9) gives

$$\langle\mu,\rho(t)\rangle \leq \limsup_{n\to\infty}\delta^*(\mu,G(t,v_n(\theta_n(t)),u_n(\theta_n(t))),$$

since $\delta^*(\mu,G(t,\cdot,\cdot))$ is upper semicontinuous on $[T_0,T]\times\mathbf{R}^n\times\mathbf{R}^n$, we get

$$\langle\mu,\rho(t)\rangle \leq \delta^*(\mu,G(t,u(t),v(t))).$$

So, $d(\rho(t),G(t,u(t),v(t)))\leq 0$, and we obtain

$$\rho(t)\in G(t,u(t),v(t)) \text{ a.e. } \in [T_0,T].$$

Consequently $-\dot{u}(t)\in N_{K(t,v(t))}(u(t))+G(t,v(t),u(t))+g(t,v(t),u(t))$. This completes the proof of the theorem.

When $\int_{T_0}^{T}\beta(s)ds > \frac{1-L}{4(1+T)}$, we subdivide $[T_0,T]$ into intervals satisfying (2) and thanks to the foregoing, the problem $(\mathcal{P})$ has a solution. □

## 4. Conclusion

In this work, we studied the existence of solution to a second order perturbed state-dependent non-convex sweeping process. The existence is established without use of the classical approaches of fixed point theory and catching up algorithm. In addition, we consider a general unbounded perturbation as a sum of a single valued and set-valued mappings, we weaken the hypotheses on the perturbation by taking a Carathéodory mapping satisfying a linear growth condition and an unbounded set-valued perturbation for which only the element of minimum norm satisfies a linear growth condition. Note that unboundedness of values of set-valued mappings is a quite natural property in the optimal control theory, see [**26**]. This result, stated for nonconvex uniformly r-prox regular sets, can be extended to a more large class of sets, namely, subsmooth sets in a Hilbert space. A natural question would be an application to optimal control and relaxation problems. Both studies will be the subject of forthcoming research projects.

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## References

[1] S. Adly and B. K. Le, *Unbounded second-order state-dependen Moreau's sweeping processes in Hilbert spaces,* J. Optim. Theory Appl., **169**(2)(2016), 407-423.

[2] D. Affane, M. Aissous and M. F. Yarou, *Existence results for sweeping process with almost convex perturbation.* Bull. Math. Soc. Sci. Math. Roumanie, **61**(109)(2018), 119-134.

[3] D. Affane, M. Aissous and M. F. Yarou, *Almost mixed semi-continuous perturbation of Moreau's sweeping process,* Evol. Equ. Control Theory, **9**(1)(2020), 27-38.

[4] D. Affane and L. Boulkemh, *Topological properties for a perturbed first order sweeping process,* Acta Univ. Sapientiae Math., **13**(1)(2021), 1–22.

[5] D. Affane and M. F. Yarou, *Unbounded perturbation for a class of variational inequalities,* Discuss. Math., Diff. Inclus., Control Optim., **37**(2017), 83-99.

[6] D. Affane and M. F. Yarou, *Second order sweeping process with almost convex perturbation,* AIP Conf. Proc. (2019), 2183, 040002. https://doi.org/10.1063/1.5136122

[7] D. Affane and M. F. Yarou, *General second order functional differential inclusion driven by sweeping process with subsmooth sets,* J. Nonlinear Funct. Anal., 2020(2020), 1-18, Article ID 26.

[8] D. Affane and M.F. Yarou, *Second-order perturbed state-dependent sweeping process with subsmooth sets,* Comput. Math. Appl., 2020(2020) 147-169.

[9] D. Affane and M.F. Yarou, *Perturbed first-order state dependent Moreau's sweeping process,* Int. J. Nonlinear Anal. Appl., **12**(2021), 605-615

[10] J. P. Aubin and A. Cellina, *Differential inclusions*, Springer-Verlag, New York, 1984.

[11] S. Boudada and M. F. Yarou, *Sweeping process with right uniformly lower semicontinuousmappings,* Positivity, 24(2020), 207-228.

[12] M. Bounkhel and L. Thibault, *Nonconvex sweeping process and prox-regularity in Hilbert space,* J. Nonlin. Convex Anal., **6**(2)(2005), 359-374.

[13] M. Bounkhel and M. F. Yarou, *Existence results for first and second order nonconvex sweeping process with delay,* Portugaliae Math., **61**(2)(2004), 2007-2030.

[14] C. Castaing, *Quelques problèmes d'évolution du second ordre,* Sem. Anal. Convexe, Montpellier, Exposé No 5 (1988).

[15] C. Castaing C., A. G. Ibrahim and M. F. Yarou, *Existence problems in second order evolution inclusions: discretization and variational approach,* Taiwanese J. Math., **12**(6)(2008), 1435-1477.

[16] C. Castaing C., A. G. Ibrahim and M. F. Yarou, *Some contributions to nonconvex sweeping process,* J. Nonlin. Convex Anal., **10**(2009), 1-20.

[17] N. Chemetov and M. D. P. Monteiro Marques, *Non-convex Quasi-variational Differential Inclusions,* Set-Valued Variat. Anal., **5**(3)(2007), 209-221.

[18] K. Chraibi, *Resolution de probleme de rafle et application a un probleme de frotement,* Topol. Meth. Nonlin. Anal., **18**(2001), 89-102.

[19] F. Clarke, R. J. Stern and P. R. Wolneski, *Proximal smoothness and the lower $C^2$ property,* J. Convex Anal., **2**(1995), 117-144.

[20] F. Clarke, Y. S. Ledyaev, R. J. Stern and P. R. Wolneski, *Nonsmooth analysis and control theory,* Springer, New York, 1998.

[21] J. F. Edmond and L. Thibault, *Relaxation of an optimal control problem involving a perturbed sweeping processes,* Math. Program. Ser. B., **104**(2005), 347-373.

[22] T. Haddad, E. Kecis and L. Thibault, *Reduction of state dependent sweeping process to unconstrained differential inclusion,* J. Glob. Optim., **62** (2015), 167-182.

[23] A. Jourani and E. Vilches, *Moreau-Yosida regularization of state-dependent sweeping processes with nonregular sets,* J. Optim. Theory Appl., **173**(1)(2017), 91-116.

[24] S. Lounis, T. Haddad and M. Sene, *Non-convex second-order Moreau's sweeping processes in Hilbert spaces,* J. Fixed Point Theory Appl., **19**(4)(2017), 2895-2908.

[25] M. Kunze and M. D. P. Monteiro Marques, *On parabolic quasi-variational inequalities and state-dependent sweeping processes,* Topol. Meth. Nonlin. Anal., **12**(1998), 179-191.

[26] P. D. Loewen and R. T. Rockafellar, *Optimal control of unbounded differential inclusions,* SIAM J. Control Optim., **32**(2)(1994), 442-470.

[27] J. J. Moreau, *Evolution problem asssociated with a moving convex set in a Hilbert space,* J. Diff. Equs., **26**(1977), 347-374.

[28] J. Noel, *Second-order general perturbed sweeping process differential inclusion,* J. Fixed Point Theory Appl., **20**(3)(2018), 133, 1-21.

[29] R. A. Poliquin, R. T. Rockafellar and L. Thibault, *Local differentiability of distance functions.* Trans. Amer. Math. Soc., **352**(11)(2000), 5231–5249.

[30] V. Recupero, *Sweeping processes and rate independence,* J. Convex Anal., **23**(2016), 921-946.

[31] L. Thibault, *Moreau sweeping process with bounded truncated retraction,* J. Convex Anal., **23**(2016), 1051-1098.

[32] A. A. Tolstonogov, *Sweeping process with unbounded nonconvex perturbation,* Nonlin. Anal., **108**(2014), 291-301.

[33] E. Vilches, *Regularization of perturbed state-dependent sweeping processes with nonregular sets,* J. Nonlin. Convex Anal., **19(4)**(2018), 633-651.

[34] M. F. Yarou, *Reduction approach to second order perturbed state-dependent sweeping process,* Creat. Math. Inform., **28**(02)(2019), 215-221.

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