Mathematical Analysis and its Contemporary Applications
Volume 3, Issue 3, 2021, 27–32
doi: 10.30495/maca.2021.1929965.1011
ISSN 2716-9898

A new version of the Hahn-Banach theorem in b-Banach spaces

Mohammad Reza Haddadi, Hossein Alaeidizaji, and Vahid Parvaneh

Abstract. In this paper, we introduce the notion of b-Banach spaces and we present some examples. Also, we give an important extension of the Hahn-Banach theorem in a b-Banach space with an application.

1. Introduction and Preliminaries

A Banach space is a complete normed vector space. A Banach space is thus a vector space with a metric that allows the calculation of vector length and distance between vectors, and it is complete in the sense that a Cauchy sequence of vectors always converges to a unique limit within the space.

Consistent with [1] and [3] the following definitions we will discuss on b-Banach spaces.

The characterization of b-metrics is their discontinuity in general. So, as a b-norm generates a b-metric, it is not continues in general. Measure of noncompactness and its weak version [4], also, PPf dependent fixed point results [2] which are applied in studying the delay integral equations, delay fractional integral equations and other related topics are subjects due to Banach spaces.Therefore, via studying the b-Banach spaces, one can also demonstrate this equations in this new structure.

In this section, we introduce the concept of b-Banach space and we present some examples. We start by definition of a b-norm function.

Definition 1.1. *Let X be a vector space and $s \geq 1$ be a given real number. A function $\|\cdot\| : X \to [0, +\infty)$ is a b-norm iff, for all $x, y \in X$, the following conditions are satisfied:*

b_1. $\|x\| = 0$ *iff* $x = 0$,

b_2. $\|\lambda x\| = |\lambda|^s \|x\|$,

2010 *Mathematics Subject Classification.* Primary: b-normed space, b-Banach Space; Secondary: Hahn-Banach theorem.

Key words and phrases. Write your keywords.

b_3. $\|x+y\| \le s(\|x\|+\|y\|)$.
The pair $(X, \|\cdot\|)$ is called a b-normed space.

Definition 1.2. *A b-complete b-normed space is called a b-Banach space.*

Here, we present an example to show that in general a b-normed need not necessarily be a norm.

Example 1.3. *Let $(X, \|\cdot\|)$ be a normed space, and $\|x\|_* = \|x\|^p$. Then $\|.\|_*$ is a b-normed. For example, for $X = \mathbb{R}$, $f(x) = |x|^p$ is a b-norm on $\mathbb{R}$ with $s = 2^{p-1}$, but is not a norm on $\mathbb{R}$.*

Example 1.4. *Let X be the set of all Lebesgue measurable functions on $[0,1]$ such that*

$$\int_0^1 |f(x)|^2\,dx < +\infty.$$

Define $\|f\| = \sqrt{\int_0^1 |f(x)|^2\,dx}$ which is a norm on X. Then, from the previous example, $\|.\|^2$ is a b-norm on X, with $s = 2$.

Every b-normed space $(X, \|.\|)$ is a b-metric space (X, d) with the induced b-metric $d(x,y) = \|x-y\|$.

Let $s \ge 1$ be a given real number. In every vector space X, we can easily define a function

$$d(x,y) = \begin{cases} 0, & x = y, \\ \\ 2^s, & x \neq y, \end{cases}$$

which is a b-metric on X which is not necessarily a b-normed space, normed space and metric space.

Definition 1.5. *A b-Banach space X is said to be b-strictly convex if*

$$\|x+y\| < s(\|x\|+\|y\|),$$

for all $x, y \in X$ with $x \neq y$.

Example 1.6. *Consider $X = \mathbb{R}^n$ $(n \neq 2)$ with a norm $\|\cdot\|$ defined by*

$$\|x\| = \max_{1\le i\le n}\{x_i^2\}, \;\; x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

Then X is a b-normed space with $s = 2$ which is not b-strictly convex. To see it, let $x = (1,0,0,...,0)$ and $y = (1,1,0,...,0)$. It is easy to see that $x \neq y$, $\|x\| = \|y\| = 1$, and $\|x+y\| = 4 = 2(\|x\|+\|y\|)$.

Example 1.7. *Consider $X = \mathbb{R}^n$ $(n \neq 2)$ with a norm $\|.\|$ defined by*

$$\|x\| = \sum_{i=1}^{n} x_i^2, \ x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

Then X is a b-normed space with $s = 2$ which is not b-strictly convex. To see it, let $x = (1, 0, ..., 0)$ and $y = (1, 0, ..., 0)$. It is easy to see that $x \neq y$, $\|x\| = \|y\| = 1$, and $\|x + y\| = 4 = 2(\|x\| + \|y\|)$.

In the following, the results are straightforward derived from the definition.

Theorem 1.8. *Let X and Y be b-normed spaces and let $T : X \to Y$ be a linear operator. Then the following are equivalent.*
(i) The operator T is b-continuous.
(ii) The operator T is continuous at 0.
(iii) The operator T is b-bounded on X.

2. Some results

Let $B^b(X, Y)$ consists of all b-bounded linear operators from a b-normed space X into a b-normed space Y. Also, for each $T \in B^b(X, Y)$ the b-norm of T is the nonnegative real number

$$\sup\{\|Tx\| : x \in B_X\}.$$

Since $\|x\|^{-s}\|Tx\| = \|T(\frac{x}{\|x\|})\| \leq \|T\|$, hence $\|Tx\| \leq \|T\|\|x\|^s$, for all $x \in X$.

Theorem 2.1. *Let X and Y be b-normed spaces. Then $B^b(X, Y)$ is a normed space under the operator norm. If Y be a b-Banach space, then so is $B^b(X, Y)$ is a b-Banach space.*

Proof. Suppose that $T_1, T_2 \in B^b(X, Y)$. It is clear that $\|T_1\| \geq 0$. Then there is an $x_0 \in X$, necessarily nonzero, such that $T_1 x_0 \neq 0$, and so $T_1(\frac{x_0}{\|x_0\|}) \neq 0$. It follows that $T_1 = 0$ if and only if $T_1 x = 0$ for each $x \in X$, that is, if and only if $\|T_1\| = 0$. If λ be a scalar, then

$$\|\lambda T_1\| = sup\{\|\lambda T_1(x)\| : x \in B_X\} = |\lambda|^s sup\{\|T_1(x)\| : x \in B_X\} = |\lambda|^s \|T_1\|.$$

If $x_0 \in B_X$, then

$$\|(T_1 + T_2)(x_0)\| \leq s(\|T_1\|\|x_0\|^s + \|T_2\|\|x_0\|^s) \leq s(\|T_1\| + \|T_2\|),$$

and so

$$\|T_1 + T_2\| = \sup\{\|(T_1 + T_2)(x_0)\| : x \in B_X\} \leq s(\|T_1\| + \|T_2\|).$$

Thus, the operator norm is a b-norm on $B^b(X, Y)$. Suppose that Y is a b-Banach space. Let $\{T_n\}$ be a Cauchy sequence in $B^b(X, Y)$. If $x \in X$, then for all $n, m \in \mathbb{N}$ we have

$$\|T_n x - T_m x\| = \|(T_n - T_m)(x)\| \leq \|T_n - T_m\|\|x\|^s.$$

It follows that the sequence $\{T_n\}$ is Cauchy in Y and hence convergent. Define $T : X \to Y$ by $Tx = \lim_n T_n x$. Since the vector space Y is b-continuous, the map T is linear. To see that T is b-bounded, first notice that the boundedness of the Cauchy sequence $\{T_n\}$ gives a $M > 0$ such that $\|T\| \leq M$ for all n, so that $\|T(x)\| \leq M$ for all $x \in B_X$ and all n. Since $\lim_n \|T_n x\| \leq M$, therefore, $\|Tx\| \leq M$ for each $x \in B_X$ and so, T is b-bounded. □

The main purpose of this section is to show that a bounded linear function on a subspace of a b-Banach space can always be extended to a bounded linear functional.

Definition 2.2. *Let p be a real valued function on a vector space X. Then p is called positive homogeneous, if $p(tx) = t^s p(x)$ for some $t > 0$ and for all $x \in X$, and is called b-subadditive if $p(x + y) \leq s[p(x) + p(y)]$ whenever $x, y \in X$. If p has both properties, then it is said to be a b-subadditive homogeneous function.*

In the following theorem we give an important extension of the Hahn Banach theorem to b-Banach spaces.

Theorem 2.3. *Suppose that $p : X \to [0, +\infty)$ be b-subadditive and positive homogeneous on a vector space X, Y be a closed subspace of X such that $dim(X/Y) = n$ and f_0 be a bounded linear functional on Y such that $f_0(y) \leq p(y)$ whenever $y \in Y$. Then there is a bounded linear functional f on X such that the restriction of f to Y is f_0 and $f(x) \leq s^n p(x)$ whenever $x \in X$.*

Proof. At first, we show that if $Y \neq X$, then there is an extension f_1 of f_0 to a subspace of X larger than Y such that f_1 is still dominated by p on this subspace. Let $x_1 \in X \setminus Y$ and $Y_1 = \langle Y \cup \{x_1\} \rangle$. If $y + tx_1 = y' + t'x_1$, where $y, y' \in Y$ and $t, t' \in \mathbb{R}$, then $x_1(t - t') = y' - y \in Y$, and so $t = t'$ and $y = y'$. Thus, each member of Y_1 has a unique representation in the form $y + tx_1$, where $y \in Y$ and $t \in \mathbb{R}$. Whenever $y_1, y_2 \in Y$, since f_0 is a functional, we have

$$\begin{aligned} f_0(y_1) + f_0(y_2) &= f_0(y_1 + y_2) \\ &\leq p(y_1 - x_1 + x_1 + y_2), \\ &\leq s[p(y_1 - x_1) + p(x_1 + y_2)] \end{aligned}$$

and so,

$$f_0(y_1) - sp(y_1 - x_1) \leq sp(x_1 + y_2) - f_0(y_2).$$

It follows that

$$\sup\{f_0(y) - sp(y - x_1) : y \in Y\} \leq \inf\{sp(x_1 + y) - f_0(y) : y \in Y\}.$$

So, there is $t_1 \in \mathbb{R}$ such that

$$\sup\{f_0(y) - sp(y - x_1) : y \in Y\} \leq t_1 \leq \inf\{sp(x_1 + y) - f_0(y) : y \in Y\}.$$

Let $f_1(y+tx_1) = f_0(y)+tt_1$ for all $y \in Y$ and $t \in \mathbb{R}$. We show that f_1 is a functional on Y_1. Let $y, y' \in Y$ and $t, t' \in \mathbb{R}$. We have

$$\begin{aligned} f_1(\alpha(y+tx_1)+(y'+t'x_1)) &= f_1(\alpha y + y' + (t\alpha + t')x_1)) \\ &= f_0(\alpha y + y') + (t\alpha + t')t_1 \\ &= \alpha f_1(y+tx_1)) + f_1(y'+t'x_1). \end{aligned}$$

It follows from the definition of t_1 that for any $y \in Y$ and any positive t, we have

$$\begin{aligned} f_1(y+tx_1) = f_0(y)+tt_1 &= t[f_0(t^{-1}y)+tt_1] \\ &\le ts[p(x_1+t^{-1}y)] \\ &= sp(y+tx_1), \end{aligned}$$

and

$$\begin{aligned} f_1(y-tx_1) = f_0(y)-tt_1 &= t[f_0(t^{-1}y)-t_1] \\ &\le st[p(t^{-1}y-x_1)] \\ &= sp(y-tx_1), \end{aligned}$$

that is, for all $x \in Y_1$, we have $f_1(x) \le sp(x)$. □

Theorem 2.4. *Let Y be a closed subspace of a b-normed space X such that $dim(X/Y) = n$ and T_0 be a bounded functional on Y. Then T_0 can be extended to a bounded functional T defined on X such that $\|T_0\| \le \|T\| \le s^n\|T_0\|$.*

Proof. Let $p(x) = \|T_0\|\|x\|^s$, for any $x \in X$. Thus, p is b-subadditive and positive homogeneous on X and $T_0(x) \le p(x)$ for all $x \in Y$. By Theorem 2.3 and its proof, there is a real positive extension T of T_0 defined on X such that for all $x \in X$,

$$\|Tx\| \le s^n p(x),$$

and so, for all $x \in X$, we have

$$\|Tx\| \le s^n\|T_0\|\|x\|^s,$$

and so, $\|T\| \le s^n\|T_0\|$. □

References

[1] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., **30**(1989), 26–37.

[2] S.R. Bernfeld, V. Lakshmikantham and Y.M. Reddy, *Fixed point theorems of operators with PPF dependence in Banach spaces*, Applicable Anal., **6**(1977), 271–280.

[3] S. Czerwik, *Nonlinear set-valued contraction mappings in b−metric spaces*, Atti Sem. Mat. Fis. Univ. Modena., **46**(2)(1998), 263-276.

[4] Kuratowski, K. *Sur les espaces complets*. Fund. Math., **15**(1930), 301–309.

Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
Email address: haddadi@abru.ac.ir

Department of Mathematics, Payame Noor University, P.O. Box. 19395-3697, Tehran, Iran
Email address: alaeidizaj.hossein@gmail.com

Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran
Email address: zam.dalahoo@gmail.com,

Received : *May* 2021
Accepted : *June* 2021