Mathematical Analysis and its Contemporary Applications
Volume 3, Issue 1, 2021, 13-31
doi: 10.30495/MACA.2021.680135

Generalized Ulam-Hyers stability of an alternate additive-quadratic-quartic functional equation in fuzzy Banach spaces

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Abstract. In this paper, we obtain and establish the generalized Ulam-Hyers stability of an additive-quadratic-quartic functional equation in fuzzy Banach spaces.

1. Introduction and Preliminaries

For the past eight years the stability of functional equations was a hot topic in this research field. The first stability question was raised by S. M. Ulam [39] in 1940.

For very general functional equations, the concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? If the answer is affirmative, we would say that the equation is stable.

The first affirmative partial answer to the question of Ulam for Banach spaces was given by D. H. Hyers [21] in the succeeding year 1941. It was further generalized and admirable outcome was achieved by a number of authors [2, 17, 30, 31, 37, 38, 40]. The solution and generalized Ulam-Hyers of various additive, quadratic, cubic and quartic functional equations in a variety of Banach spaces was established in [1, 6, 12, 13, 14, 18, 19, 23, 29, 33, 32, 34, 35], and references cited therein.

2010 *Mathematics Subject Classification.* Primary: 39B52, 39B72; Secondary: 39B82.

Key words and phrases. Additive functional equations, Quadratic functional equations, Quartic functional equations, Mixed type functional equations, Ulam-Hyers-Rassias stability, Fuzzy Banach Space.

The general solution and generalized Ulam-Hyers stability of additive-quadratic-quartic functional equations

$$\begin{aligned} f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z) \\ = 2\left[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)\right] \\ -4f(x)-4f(y)-2\left[f(z)+f(-z)\right] \end{aligned} \tag{1}$$

using Hyers direct method was investigated by J. M. Rassias et. al., in [36]. The following elementary results are taken from [36].

Lemma 1.1. [36] *If $f:\mathcal{G}\to\mathcal{H}$ is an odd mapping satisfying (1), then*

$$f(2x)=2f(x) \tag{2}$$

for all $x\in\mathcal{G}$, where f is additive.

Lemma 1.2. [36] *If $f:\mathcal{G}\to\mathcal{H}$ is an even mapping satisfying (1) and if $q_2:\mathcal{G}\to\mathcal{H}$ is a mapping given by*

$$q_2(x)=f(2x)-16f(x) \tag{3}$$

for all $x\in\mathcal{G}$, then

$$q_2(2x)=4q_2(x) \tag{4}$$

for all $x\in\mathcal{G}$, where q_2 is quadratic.

Lemma 1.3. [36] *If $f:\mathcal{G}\to\mathcal{H}$ is an even mapping satisfying (1) and if $q_4:\mathcal{G}\to\mathcal{H}$ is a mapping given by*

$$q_4(x)=f(2x)-4f(x) \tag{5}$$

for all $x\in\mathcal{G}$, then

$$q_4(2x)=16q_4(x) \tag{6}$$

for all $x\in\mathcal{G}$, where q_4 is quartic.

Remark 1.4. [36] *Let $f:\mathcal{G}\to\mathcal{H}$ be a mapping satisfying (1) and $q_2,q_4:\mathcal{G}\to\mathcal{H}$ be a mapping defined in (3) and (5) then*

$$f(x)=\frac{1}{12}(q_4(x)-q_2(x)) \tag{7}$$

for all $x\in\mathcal{G}$.

In this paper, we prove the generalized Ulam-Hyers stability of the additive-quadratic-quartic functional equation (1) in fuzzy Banach spaces. It is easy to check that equation (1) has a solution as $g(x)=ax+bx^2+cx^4$.

2. Definitions on Fuzzy Banach Spaces

In this section, we present the definitions and notations on fuzzy normed spaces. We use the definition of fuzzy normed spaces given in [8, 25, 26, 27, 28]. We note that some results on the stability of various functional equations can be found in [4, 11, 15].

Definition 2.1. *Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

$(FNS1)$ $N(x,c) = 0$, *for* $c \leq 0$;

$(FNS2)$ $x = 0$ *if and only if* $N(x,c) = 1$, *for all* $c > 0$;

$(FNS3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

$(FNS4)$ $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;

$(FNS5)$ $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x,t) = 1$;

$(FNS6)$ *for* $x \neq 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t.

Example 2.2. *Let $(X, ||\cdot||)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 2.3. *Let (X, N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 2.4. *A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

Definition 2.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 2.6. *A mapping $f : X \longrightarrow Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$, then f is said to be continuous on X.*

The stability of a quiet number of functional equations in Fuzzy normed spaces was given in [**3, 25, 26, 27, 28**]. Throughout the paper, we consider $\mathcal{P}_3$, $(\mathcal{P}_1, N)$ and $(\mathcal{P}_2, N')$ are linear space, fuzzy normed space and fuzzy Banach space, respectively. Define a mapping $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ by

$$\begin{aligned}\mathcal{G}^1_{24}(x,y,z) = {} & g(x+y+z) + g(x+y-z) + g(x-y+z) + g(x-y-z) \\ & - 2\left[g(x+y) + g(x-y) + g(y+z) + g(y-z) + g(x+z) + g(x-z)\right] \\ & - 4g(x) - 4g(y) - 2\left[g(z) + g(-z)\right]\end{aligned}$$

for all $x, y, z \in \mathcal{P}_1$.

3. Fuzzy Stability Results: Additive Case

In this section, we investigate the stability of (1) in the odd case.

Theorem 3.1. *Let $a = \pm 1$ and $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be an odd mapping satisfying the functional inequality*

$$N\left(\mathcal{G}^1_{24}(x,y,z), s\right) \geq N'\left(\omega\left(x,y,z\right), s\right) \tag{8}$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$, where $\omega, \Omega : \mathcal{P}_1^2 \longrightarrow \mathcal{P}_3$ be a mapping with the conditions

$$\lim_{b\to\infty} N'\left(\omega\left(2^{ab}x, 2^{ab}y, 2^{ab}z\right), 2^{ab}s\right) = 1 \tag{9}$$

and

$$N'\left(\Omega_A\left(2^a y\right), s\right) \geq N'\left(c^a \Omega_A\left(y\right), s\right) \tag{10}$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$, for some $c > 0$ with $0 < \left(\frac{c}{2}\right)^a < 1$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ which satisfies (1) and the inequality

$$N\left(g_a(y) - \mathcal{A}(y), s\right) \geq N'\left(\Omega_A\left(y\right), \frac{s|2-c|}{4}\right) \tag{11}$$

where $\Omega_A\left(y\right)$ and $\mathcal{A}(y)$ are defined by

$$N'\left(\Omega_A(y), s\right) = \min\left\{N'\left(\omega\left(y,y,y\right), s\right), N'\left(\omega\left(-y,y,y\right), s\right), \right\} \tag{12}$$

and

$$\lim_{b\to\infty} N\left(\mathcal{A}(y) - \frac{g_a(2^{ab}y)}{2^{ab}}, s\right) = 1 \tag{13}$$

for all $y \in \mathcal{P}_1$ and $s > 0$, respectively.

Proof. Replacing (x, y, z) by (y, y, y) in (8) and using oddness of g_a, we arrive

$$N\left(g_a(3y) - 6g_a(2y) + 9g_a(y), s\right) \geq N'\left(\omega\left(y,y,y\right), s\right) \tag{14}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Setting (x, y, z) by $(-y, y, y)$ in (8) and using oddness of g_a, we obtain

$$N\left(-g_a(3y) + 2g_a(2y) - g_a(y), s\right) \geq N'\left(\omega\left(-y,y,y\right), s\right) \tag{15}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. It follows from (14), (15) and $(FNS4)$, we get

$$\begin{aligned}
&N\left(8g_a(y) - 4g_a(2y), 2s\right)\\
&= N\left(g_a(3y) - 6g_a(2y) + 9g_a(y) - g_a(3y) + 2g_a(2y) - g_a(y), s + s\right)\\
&\geq \min\left\{N\left(g_a(3y) - 6g_a(2y) + 9g_a(y), s\right), N\left(-g_a(3y) + 2g_a(2y) - g_a(y), s\right)\right\}\\
&\geq \min\left\{N'\left(\omega\left(y, y, y\right), s\right), N'\left(\omega\left(-y, y, y\right), s\right)\right\} = N'\left(\Omega_A\left(y\right), s\right) \qquad (16)
\end{aligned}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Using $(FNS3)$ in (16), we have

$$N\left(\frac{g_a(2y)}{2} - g_a(y), \frac{s}{4}\right) \geq N'\left(\Omega_A\left(y\right), s\right) \tag{17}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Replacing y by $2^b y$ in (17), we obtain

$$N\left(\frac{g_a(2^{b+1}y)}{2} - g_a(2^b y), \frac{s}{4}\right) \geq N'\left(\Omega_A\left(2^b y\right), s\right) \tag{18}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Using (10), $(FNS3)$ in (18), we arrive

$$N\left(\frac{g_a(2^{b+1}y)}{2} - g_a(2^b y), \frac{s}{4}\right) \geq N'\left(\Omega_A\left(y\right), \frac{s}{c^b}\right) \tag{19}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. With the help of $(FNS3)$ it follows from (19), that

$$N\left(\frac{g_a(2^{b+1}y)}{2^{(b+1)}} - \frac{g_a(2^b y)}{2^b}, \frac{s}{4 \cdot 2^b}\right) \geq N'\left(\Omega_A\left(y\right), \frac{s}{c^b}\right) \tag{20}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Changing s by $c^b s$ in (20), we get

$$N\left(\frac{g_a(2^{b+1}y)}{2^{(b+1)}} - \frac{g_a(2^b y)}{2^b}, \frac{s}{4} \cdot \left(\frac{c}{2}\right)^b\right) \geq N'\left(\Omega_A\left(y\right), s\right) \tag{21}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. It is easy to see that

$$\frac{g_a(2^b y)}{2^b} - g_a(y) = \sum_{d=0}^{b-1}\left[\frac{g_a(2^{d+1}y)}{2^{(d+1)}} - \frac{g_a(2^d y)}{2^d}\right] \tag{22}$$

for all $y \in \mathcal{P}_1$. From equations (21) and (22), we have

$$\begin{aligned}
&N\left(\frac{g_a(2^b y)}{2^b} - g_a(y), \frac{s}{4} \cdot \sum_{d=0}^{b-1}\left(\frac{c}{2}\right)^d\right)\\
&\qquad\geq \min \bigcup_{d=0}^{b-1}\left\{N\left(\frac{g_a(2^{d+1}y)}{2^{(d+1)}} - \frac{g_a(2^d y)}{2^d}, \frac{s}{4} \cdot \left(\frac{c}{2}\right)^d\right)\right\}\\
&\qquad\geq \min \bigcup_{d=0}^{b-1}\left\{N'\left(\Omega_A\left(y\right), s\right)\right\}\\
&\qquad= N'\left(\Omega_A\left(y\right), s\right) \qquad (23)
\end{aligned}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Replacing y by $2^e x$ in (23) and using (10), $(FNS3)$, and substituting s by $c^e s$, we obtain

$$N\left(\frac{g_a(2^{b+e}y)}{2^{(b+e)}} - \frac{g_a(2^e y)}{2^e}, \frac{s}{4} \cdot \sum_{d=e}^{b+e-1} \left(\frac{c}{2}\right)^d\right) \geq N'\left(\Omega_A\left(y\right), s\right) \tag{24}$$

for all $y \in \mathcal{P}_1$, $s > 0$ and $e > b \geq 0$. It follows from (24), we find

$$N\left(\frac{g_a(2^{b+e}y)}{2^{(b+e)}} - \frac{g_a(2^e y)}{2^e}, s\right) \geq N'\left(\Omega_A\left(y\right), \frac{s}{\frac{1}{4} \cdot \sum_{d=e}^{b+e-1} \left(\frac{c}{2}\right)^d}\right) \tag{25}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Since $0 < t < 2$ and $\displaystyle\sum_{d=0}^{b} \left(\frac{c}{2}\right)^d < \infty$, the Cauchy criterion for convergence and $(FNS5)$ implies that $\left\{\dfrac{g_a(2^b y)}{2^b}\right\}$ is a Cauchy sequence in $(\mathcal{P}_2, N')$. Since $(\mathcal{P}_2, N')$ is a fuzzy Banach space, this sequence converges to some point $\mathcal{A} \in \mathcal{P}_2$. So one can define the mapping $\mathcal{A} : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ by

$$\lim_{q \to \infty} N\left(\mathcal{A}(y) - \frac{g_a(2^b y)}{2^b}, s\right) = 1 \tag{26}$$

for all $y \in \mathcal{P}_1$ and all $s > 0$. Letting $e = 0$ and $b \to \infty$ in (25), we get

$$N\left(\mathcal{A}(y) - g_a(y), s\right) \geq N'\left(\Omega_A\left(y\right), \frac{s(2\ - c)}{4}\right)$$

for all $y \in \mathcal{P}_1$ and all $s > 0$. To prove $\mathcal{A}$ satisfies the (1), replacing (x, y, z) by $(2^b x, 2^b y, 2^b z)$ in (10), we obtain

$$\begin{aligned} N\left(D\mathcal{A}(x, y, z), s\right) &= N\left(\frac{1}{2^b}\mathcal{G}_{24}^1\left(2^b x, 2^b y, 2^b z\right), s\right) \\ &\geq N'\left(\omega\left(2^b x, 2^b y, 2^b z\right), 2^b s\right) \end{aligned} \tag{27}$$

for all $x, y, z \in \mathcal{P}_1$ and all $s > 0$. Now,

$$
\begin{aligned}
&N\left(\mathcal{A}(x+y+z)+\mathcal{A}(x+y-z)+\mathcal{A}(x-y+z)+\mathcal{A}(x-y-z)\right.\\
&\quad -2\left[\mathcal{A}(x+y)+\mathcal{A}(x-y)+\mathcal{A}(y+z)+\mathcal{A}(y-z)+\mathcal{A}(x+z)+\mathcal{A}(x-z)\right]\\
&\qquad \left.-4\mathcal{A}(x)-4\mathcal{A}(y)-2\left[\mathcal{A}(z)+\mathcal{A}(-z)\right], s\right),\\
&\geq \min\left\{N\left(\mathcal{A}(x+y+z)-\frac{g_a(2^b(x+y+z))}{2^b}, \frac{s}{15}\right),\right.\\
&\qquad N\left(\mathcal{A}(x+y-z)-\frac{g_a(2^b(x+y-z))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(\mathcal{A}(x-y+z)-\frac{g_a(2^b(x-y+z))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(\mathcal{A}(x-y-z)-\frac{g_a(2^b(x-y-z))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(-2\mathcal{A}(x+y)+\frac{2g_a(2^b(x+y))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(-2\mathcal{A}(x-y)+\frac{2g_a(2^b(x-y))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(-2\mathcal{A}(y+z)+\frac{2g_a(2^b(y+z))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(-2\mathcal{A}(y-z)+\frac{2g_a(2^b(y-z))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(-2\mathcal{A}(x+z)+\frac{2g_a(2^b(x+z))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(-2\mathcal{A}(x-z)+\frac{2g_a(2^b(x-z))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(-4\mathcal{A}(x)+\frac{4g_a(2^bx)}{2^b}, \frac{s}{15}\right), N\left(-4\mathcal{A}(y)+\frac{4g_a(2^by)}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(-2\mathcal{A}(z)+\frac{2g_a(2^bz)}{2^b}, \frac{s}{15}\right), N\left(-2\mathcal{A}(-z)+\frac{2g_a(2^b(-z))}{2^b}, \frac{s}{15}\right),\\
&\qquad N\left(\frac{g_a(2^b(x+y+z))}{2^b}+\frac{g_a(2^b(x+y-z))}{2^b}+\frac{g_a(2^b(x-y+z))}{2^b}\right.\\
&\qquad\quad +\frac{g_a(2^b(x-y-z))}{2^b}-\frac{2g_a(2^b(x+y))}{2^b}-\frac{2g_a(2^b(x-y))}{2^b}\\
&\qquad\quad -\frac{2g_a(2^b(y+z))}{2^b}-\frac{2g_a(2^b(y-z))}{2^b}-\frac{2g_a(2^b(x+z))}{2^b}\\
&\qquad\quad -\frac{2g_a(2^b(x-z))}{2^b}-\frac{4g_a(2^bx)}{2^b}-\frac{4g_a(2^by)}{2^b}-\frac{2g_a(2^b(z))}{2^b}\\
&\qquad\quad \left.\left.-\frac{2g_a(2^b(-z))}{2^b}, \frac{s}{15}\right)\right\}
\end{aligned}
\tag{28}
$$

for all $x, y, z \in \mathcal{P}_1$ and all $s > 0$. Using (26), (27) and $(FNS5)$ in (28), we reach

$$\begin{aligned} & N\left(\mathcal{A}(x+y+z)+\mathcal{A}(x+y-z)+\mathcal{A}(x-y+z)+\mathcal{A}(x-y-z)\right. \\ & \quad -2\left[\mathcal{A}(x+y)+\mathcal{A}(x-y)+\mathcal{A}(y+z)+\mathcal{A}(y-z)+\mathcal{A}(x+z)+\mathcal{A}(x-z)\right] \\ & \quad \left. -4\mathcal{A}(x)-4\mathcal{A}(y)-2\left[\mathcal{A}(z)+\mathcal{A}(-z)\right], s\right) \\ & \qquad \geq \min\left\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1, N'\left(\omega\left(2^b x, 2^b y, 2^b z\right), 2^b s\right)\right\} \end{aligned} \tag{29}$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$. Approaching b tends to infinity in (29) and applying (10), we get

$$\begin{aligned} & N\left(\mathcal{A}(x+y+z)+\mathcal{A}(x+y-z)+\mathcal{A}(x-y+z)+\mathcal{A}(x-y-z)\right. \\ & \quad -2\left[\mathcal{A}(x+y)+\mathcal{A}(x-y)+\mathcal{A}(y+z)+\mathcal{A}(y-z)+\mathcal{A}(x+z)+\mathcal{A}(x-z)\right] \\ & \quad \left. -4\mathcal{A}(x)-4\mathcal{A}(y)-2\left[\mathcal{A}(z)+\mathcal{A}(-z)\right], s\right) = 1 \end{aligned} \tag{30}$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$. Using $(FNS2)$ in (30), we see that

$$\begin{aligned} & \mathcal{A}(x+y+z)+\mathcal{A}(x+y-z)+\mathcal{A}(x-y+z)+\mathcal{A}(x-y-z) \\ & \quad = 2\left[\mathcal{A}(x+y)+\mathcal{A}(x-y)+\mathcal{A}(y+z)+\mathcal{A}(y-z)+\mathcal{A}(x+z)+\mathcal{A}(x-z)\right] \\ & \qquad + 4\mathcal{A}(x)-4\mathcal{A}(y)+2\left[\mathcal{A}(z)+\mathcal{A}(-z)\right] \end{aligned}$$

for all $x, y, z \in \mathcal{P}_1$. Hence $\mathcal{A}$ satisfies the functional equation (1). To prove $\mathcal{A}(y)$ is unique, let $\mathcal{A}'(y)$ be another additive functional equation satisfying (1) and (13). Thus,

$$\begin{aligned} N(\mathcal{A}(y)-\mathcal{A}'(y), s) &= N\left(\frac{\mathcal{A}(2^b y)}{2^b} - \frac{\mathcal{A}'(2^b y)}{2^b}, s\right) \\ &\geq \min\left\{N\left(\frac{\mathcal{A}(2^b y)}{2^b} - \frac{g_a(2^b y)}{2^b}, \frac{s}{2}\right), N\left(\frac{\mathcal{A}'(2^b y)}{2^b} - \frac{g_a(2^b y)}{2^b}, \frac{s}{2}\right)\right\} \\ &\geq N'\left(\Omega_A\left(2^b y\right), \frac{s(2-c)2^b}{2 \cdot 4}\right) \\ &= N'\left(\Omega_A\left(y\right), \frac{s(2-c)2^b}{2\ c^b\ 4}\right) \end{aligned}$$

for all $y \in \mathcal{P}_1$ and all $s > 0$. Since

$$\lim_{q \to \infty} \frac{s(2-c)2^b}{2\ c^b\ 4} = \infty,$$

we obtain

$$\lim_{q \to \infty} N'\left(\Omega_A\left(y\right), \frac{s(2^{16}-c)2^b}{2\ c^b\ 4}\right) = 1$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Thus

$$N(\mathcal{A}(y)-\mathcal{A}'(y), s) = 1$$

for all $y \in \mathcal{P}_1$ and $s > 0$, and so $\mathcal{A}(y) = \mathcal{A}'(y)$. Therefore $\mathcal{A}(y) - \mathcal{A}'(y)$ is unique. Hence for $a = 1$ the theorem holds. Replacing y by $\frac{y}{2}$ in (16), we arrive

$$N\left(g_a(y) - 2f\left(\frac{y}{2}\right), \frac{s}{4}\right) \geq N'\left(\Omega_A\left(\frac{y}{2}\right), s\right) \tag{31}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. The rest of the proof is similar lines to that of case $a = 1$ Hence the theorem holds for the case $a = -1$. This completes the proof. □

The following corollary is the immediate consequence of Theorem 3.1 concerning the stabilities of (1).

Corollary 3.2. *Assume q and r be positive numbers. Let $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be an odd mapping satisfying the functional inequality*

$$N\left(\mathcal{G}_{24}^1(x, y, z), s\right) \geq \begin{cases} N(q, s) \\ N\left(q\left(||x||^r + ||y||^r + ||z||^r\right), s\right), \\ N\left(q\ ||x||^r||y||^r||z||^r, s\right), \\ N\left(q\left(||x||^{r_1} + ||y||^{r_2} + ||z||^{r_3}\right), s\right), \\ N\left(q||x||^{r_1}||y||^{r_2}||z||^{r_3}, s\right), \end{cases} \tag{32}$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$. Then there is a unique additive mapping $\mathcal{A} : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ such that

$$N\left(\mathcal{A}(y) - g_a(y), s\right) \geq \begin{cases} N'\left(8q, s|2-1|\right), & \\ N'\left(24q||y||^r, s|2-2^r|\right), & r \neq 1; \\ N'\left(8q||y||^{3r}, s|2-2^{3r}|\right), & 3r \neq 1; \\ N'\left(8q\sum_{i=1}^3 ||y||^{r_i}, \sum_{i=1}^3 s|2-2^{r_i}|\right), & r_1, r_2, r_3 \neq 1; \\ N'\left(8q||y|^{\sum_{i=1}^3 r_i}, s|2-2^{\sum_{i=1}^3 r_i}|\right), & \sum_{i=1}^3 r_i \neq 1; \end{cases} \tag{33}$$

for all $y \in \mathcal{P}_1$ and $s > 0$.

4. Fuzzy Stability Results: Quadratic Case

In this section, we investigate the stability of (1) in quadratic case.

Theorem 4.1. *Assume $a = \pm 1$ and $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is an even mapping satisfying the functional inequality*

$$N\left(\mathcal{G}_{24}^1(x, y, z), s\right) \geq N'\left(\omega\left(x, y, z\right), s\right) \tag{34}$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$, where $\omega, \Omega : \mathcal{P}_1^2 \longrightarrow \mathcal{P}_3$ be a mapping with the conditions

$$\lim_{b \to \infty} N'\left(\omega\left(2^{ab}x, 2^{ab}y, 2^{ab}z\right), 4^{ab}s\right) = 1 \tag{35}$$

and

$$N'\left(\Omega_Q\left(2^a y\right), s\right) \geq N'\left(c^a \Omega_Q\left(y\right), s\right) \tag{36}$$

for all $x, y, z \in \mathcal{P}_1$ and all $s > 0$, for some $c > 0$ with $0 < \left(\frac{c}{4}\right)^a < 1$. Then there exists a unique quadratic mapping $\mathcal{Q}_2 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ which satisfies (1) and the inequality

$$N\left(g_{2q}(y) - \mathcal{Q}_2(y), s\right) = N\left(g_q(2y) - 16g_q(y) - \mathcal{Q}_2(y), s\right) \geq N'\left(\Omega_Q\left(y\right), \frac{5s|4-c|}{4}\right) \tag{37}$$

where $\Omega_Q(y)$ and $\mathcal{Q}_2(y)$ are defined by

$$N'\left(\Omega_Q(y), s\right) = \min\left\{N'\left(\omega\left(y, y, y\right), s\right), N'\left(\omega\left(2y, y, y\right), s\right), \right\} \tag{38}$$

and

$$\lim_{b\to\infty} N\left(\mathcal{Q}_2(y) - \frac{g_q(2^{ab}y)}{4^{ab}}, s\right) = 1 \tag{39}$$

for all $y \in \mathcal{P}_1$ and $s > 0$, respectively.

Proof. Replacing (x, y, z) by (y, y, y) in (34) and using evenness of g_q, we arrive

$$N\left(g_q(3y) - 6g_q(2y) + 15g_q(y), s\right) \geq N'\left(\omega\left(y, y, y\right), s\right) \tag{40}$$

for all $y \in \mathcal{P}_1$ and all $s > 0$. It follows from (40) and $(FNS3)$, we get

$$N\left(4g_q(3y) - 24g_q(2y) + 60g_q(y), 4s\right) \geq N'\left(\omega\left(y, y, y\right), s\right) \tag{41}$$

for all $y \in \mathcal{P}_1$ and all $s > 0$. Setting (x, y, z) by $(-y, y, y)$ in (34) and using evenness of g_q, we obtain

$$N\left(g_q(4y) + 4g_q(2y) - 4g_q(3y) + 4g_q(y), s\right) \geq N'\left(\omega\left(-y, y, y\right), s\right) \tag{42}$$

for all $y \in \mathcal{P}_1$ and all $s > 0$. It follows from (41), (42) and $(FNS4)$, we get

$$\begin{aligned} &N\left(g_q(4y) - 20g_q(2y) + 64g_q(y), 5s\right) \\ &= N\left(4g_q(3y) - 24g_q(2y) + 60g_q(y) + g_q(4y) + 4g_q(2y) - 4g_q(3y) + 4g_q(y), 4s + s\right) \\ &\geq \min\Big\{N\left(4g_q(3y) - 24g_q(2y) + 60g_q(y), 4s\right), \\ &\qquad N\left(g_q(4y) + 4g_q(2y) - 4g_q(3y) + 4g_q(y), s\right)\Big\} \\ &\geq \min\left\{N'\left(\omega\left(y, y, y\right), s\right), N'\left(\omega\left(-y, y, y\right), s\right)\right\} = N'\left(\Omega_Q\left(y\right), s\right) \end{aligned} \tag{43}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. The above inequality can be rewritten as

$$N\left(g_q(4y) - 16g_q(2y) - 4(g_q(2y) - 16g_q(y)), 5s\right) = N'\left(\Omega_Q\left(y\right), s\right) \tag{44}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Define a mapping $g_{2q} : \mathcal{P}_1 \to \mathcal{P}_2$ by

$$g_{2q}(y) = g_q(2y) - 16g_q(y) \tag{45}$$

for all $y \in \mathcal{P}_1$. Using (45) in (44), we have

$$N\left(g_{2q}(2y) - 4g_{2q}(y), 5s\right) = N'\left(\Omega_Q\left(y\right), s\right) \tag{46}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Using $(FNS3)$ in (46), we have

$$N\left(\frac{g_{2q}(2y)}{4} - g_{2q}(y), \frac{5s}{4}\right) \geq N'\left(\Omega_Q\left(y\right), s\right) \tag{47}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. The rest of the proof is similar lines to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 4.1 concerning the stabilities of (1).

Corollary 4.2. *Assume q and r be positive numbers. Let $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be an even mapping satisfying the functional inequality*

$$N\left(\mathcal{G}_{24}^1(x,y,z), s\right) \geq \begin{cases} N\left(q, s\right) \\ N\left(q\left(||x||^r + ||y||^r + ||z||^r\right), s\right), \\ N\left(q\ ||x||^r||y||^r||z||^r, s\right), \\ N\left(q\left(||x||^{r_1} + ||y||^{r_2} + ||z||^{r_3}\right), s\right), \\ N\left(q||x||^{r_1}||y||^{r_2}||z||^{r_3}, s\right), \end{cases} \tag{48}$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$. Then there exists a unique quadratic mapping $\mathcal{Q}_2 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ such that

$$N\left(g_{2q}(y) - \mathcal{Q}_2(y), s\right) = N\left(g_q(2y) - 16g_q(y) - \mathcal{Q}_2(y), s\right)$$
$$\geq \begin{cases} N'\left(8q, 5s|4-1|\right), \\ N'\left([20 + 4\cdot 2^r]q||y||^r, 5s|4-2^r|\right), & r \neq 2; \\ N'\left([4 + 4\cdot 2^{3r}]q||y||^{3r}, 5s|4-2^{3r}|\right), & 3r \neq 2; \\ N'\left(4[1+2^{r_1}]||y||^{r_1} + 8||y||^{r_2} + 8||y||^{r_3}, \sum_{i=1}^{3} 5s|4-2^{r_i}|\right), & r_1, r_2, r_3 \neq 2; \\ N'\left(4[1+2^{r_1}]q||y|^{\sum_{i=1}^{3} r_i}, 5s|4 - 2^{\sum_{i=1}^{3} r_i}|\right), & \sum_{i=1}^{3} r_i \neq 2; \end{cases} \tag{49}$$

for all $y \in \mathcal{P}_1$ and $s > 0$.

5. Fuzzy Stability Results: Quartic Case

In this section, we prove the stability of (1) in quartic case.

Theorem 5.1. *Let $a = \pm 1$ and $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be an even mapping satisfying the functional inequality*

$$N\left(\mathcal{G}_{24}^1(x,y,z), s\right) \geq N'\left(\omega\left(x,y,z\right), s\right) \tag{50}$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$, where $\omega, \Omega : \mathcal{P}_1^2 \longrightarrow \mathcal{P}_3$ be a mapping with the conditions

$$\lim_{b\to\infty} N'\left(\omega\left(2^{ab}x, 2^{ab}y, 2^{ab}z\right), 16^{ab}s\right) = 1 \tag{51}$$

and

$$N'\left(\Omega_Q\left(2^a y\right), s\right) \geq N'\left(c^a \Omega_Q\left(y\right), s\right) \tag{52}$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$, for some $c > 0$ with $0 < \left(\frac{c}{4}\right)^a < 1$. Then there exists a unique quartic mapping $\mathcal{Q}_4 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ which satisfies (1) and the inequality

$$\begin{aligned} N\left(g_{4q}(y) - \mathcal{Q}_4(y), s\right) &= N\left(g_q(2y) - 4g_q(y) - \mathcal{Q}_4(y), s\right) \\ &\geq N'\left(\Omega_Q\left(y\right), \frac{5s|16-c|}{16}\right) \end{aligned} \tag{53}$$

where $\Omega_Q\left(y\right)$ is defined in (38) such that

$$\lim_{b\to\infty} N\left(\mathcal{Q}_4(y) - \frac{g_q(2^{ab}y)}{16^{ab}}, s\right) = 1 \tag{54}$$

for all $y \in \mathcal{P}_1$ and $s > 0$.

Proof. The inequality (43) can be rewritten as

$$N\left(g_q(4y) - 4g_q(2y) - 16(g_q(2y) - 4g_q(y)), 5s\right) = N'\left(\Omega_Q\left(y\right), s\right) \tag{55}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Define a mapping $g_{4q} : \mathcal{P}_1 \to \mathcal{P}_2$ by

$$g_{4q}(y) = g_q(2y) - 4g_q(y) \tag{56}$$

for all $y \in \mathcal{P}_1$. Using (56) in (55), we have

$$N\left(g_{4q}(2y) - 16g_{4q}(y), 5s\right) = N'\left(\Omega_Q\left(y\right), s\right) \tag{57}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Using $(FNS3)$ in (57), we have

$$N\left(\frac{g_{4q}(2y)}{16} - g_{4q}(y), \frac{5s}{16}\right) \geq N'\left(\Omega_Q\left(y\right), s\right) \tag{58}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. The rest of the proof is similar lines to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 5.1 concerning the stabilities of (1).

Corollary 5.2. *Assume q and r be positive numbers. Let $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be an even mapping satisfying the functional inequality*

$$N\left(\mathcal{G}^1_{24}(x, y, z), s\right) \geq \begin{cases} N\left(q, s\right) \\ N\left(q\left(||x||^r + ||y||^r + ||z||^r\right), s\right), \\ N\left(q\ ||x||^r||y||^r||z||^r, s\right), \\ N\left(q\left(||x||^{r_1} + ||y||^{r_2} + ||z||^{r_3}\right), s\right), \\ N\left(q||x||^{r_1}||y||^{r_2}||z||^{r_3}, s\right), \end{cases} \tag{59}$$

for all $x, y, z \in \mathcal{P}_1$ *and* $s > 0$. *Then there exists a unique quartic mapping* $\mathcal{Q}_4 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ *such that*

$$N\left(g_{4q}(y) - \mathcal{Q}_4(y), s\right) = N\left(g_q(2y) - 4g_q(y) - \mathcal{Q}_4(y), s\right)$$
$$\geq \begin{cases} N'\left(32q, 5s|16-1|\right), & \\ N'\left([20+4\cdot 2^r]q||y||^r, 5s|16-2^r|\right), & r \neq 4; \\ N'\left([80+16\cdot 2^{3r}]q||y||^{3r}, 5s|16-2^{3r}|\right), & 3r \neq 4; \\ N'\left(16[1+2^{r_1}]||y||^{r_1} + 32||y||^{r_2} + 32||y||^{r_3}, \sum_{i=1}^{3} 5s|16-2^{r_i}|\right), & r_1, r_2, r_3 \neq 4; \\ N'\left(16[1+2^{r_1}]q||y|^{\sum_{i=1}^{3} r_i}, 5s|16-2^{\sum_{i=1}^{3} r_i}|\right), & \sum_{i=1}^{3} r_i \neq 4; \end{cases} \tag{60}$$

for all $y \in \mathcal{P}_1$ *and* $s > 0$.

6. Fuzzy Stability Results: Quadratic-Quartic Case

In this section, we establish the stability of (1) in quadratic-quartic case.

Theorem 6.1. *Assume* $a = \pm 1$ *and* $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ *is an even mapping satisfying the functional inequality*

$$N\left(\mathcal{G}_{24}^1(x, y, z), s\right) \geq N'\left(\omega\left(x, y, z\right), s\right) \tag{61}$$

for all $x, y, z \in \mathcal{P}_1$ *and all* $s > 0$, *where* $\omega, \Omega : \mathcal{P}_1^2 \longrightarrow \mathcal{P}_3$ *is a mapping with conditions* (35), (36), (51) *and* (52), *for all* $x, y, z \in \mathcal{P}_1$ *and all* $s > 0$, *for some* $c > 0$ *with* $0 < \left(\frac{c}{4}\right)^a < \left(\frac{c}{16}\right)^a < 1$. *Then there exists a unique quadratic mapping* $\mathcal{Q}_2 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ *and a unique quartic mapping* $\mathcal{Q}_4 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ *which satisfies* (1) *and the inequality*

$$N\left(g_q(y) - \mathcal{Q}_2(y) - \mathcal{Q}_4(y), s\right)$$
$$\geq N'\left(\Omega_Q\left(y\right), \frac{30s|4-c|}{4}\right) + N'\left(\Omega_Q\left(y\right), \frac{30s|16-c|}{16}\right) \tag{62}$$

where $\Omega_Q\left(y\right)$ *and* $\mathcal{Q}_2(y)$ *are defined in* (38) (39) *and* (54) *for all* $y \in \mathcal{P}_1$ *and all* $s > 0$, *respectively.*

Proof. By Theorems 4.1 and 5.1, there exists a unique quadratic mapping $\mathcal{Q}_{2_1} : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ and a unique quartic mapping $\mathcal{Q}_{4_1} : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ such that

$$N\left(g_q(2y) - 16g_q(y) - \mathcal{Q}_{2_1}(y), s\right) \geq N'\left(\Omega_Q\left(y\right), \frac{5s|4-c|}{4}\right) \tag{63}$$

and

$$N\left(g_q(2y) - 4g_q(y) - \mathcal{Q}_{4_1}(y), s\right) \geq N'\left(\Omega_Q\left(y\right), \frac{5s|16-c|}{16}\right) \tag{64}$$

for all $y \in \mathcal{P}_1$ and $s > 0$.

Now it from (63), (64), $(FNS3)$ and $(FNS4)$, one can arrive

$$\begin{aligned}
&N\left(12g_q(y)+Q_{2_1}(y)-Q_{4_1}(y),2s\right)\\
&\geq N\left(g_q(2y)-16g_q(y)-\mathcal{Q}_{2_1}(y),s\right)+N\left(g_q(2y)-4g_q(y)-\mathcal{Q}_{4_1}(y),s\right)\\
&\geq N'\left(\Omega_Q\left(y\right),\frac{5s|4-c|}{4}\right)+N'\left(\Omega_Q\left(y\right),\frac{5s|16-c|}{16}\right)
\end{aligned}\tag{65}$$

for all $y\in\mathcal{P}_1$ and $s>0$. The above inequality (65) can be re-modified as

$$\begin{aligned}
&N\left(g_q(y)+\frac{1}{12}Q_{2_1}(y)-\frac{1}{12}Q_{4_1}(y),\frac{s}{6}\right)\\
&\geq N'\left(\Omega_Q\left(y\right),\frac{5s|4-c|}{4}\right)+N'\left(\Omega_Q\left(y\right),\frac{5s|16-c|}{16}\right)
\end{aligned}\tag{66}$$

for all $y\in\mathcal{P}_1$ and $s>0$.

Thus, we obtain (62) by defining $\mathcal{Q}_2(y)=\frac{-1}{12}\mathcal{Q}_{2_1}(y)$ and $\mathcal{Q}_4(y)=\frac{1}{12}\mathcal{Q}_{4_1}(y)$, where $\Omega_Q\left(y\right)$ and $\mathcal{Q}_2(y)$ are defined in (38) (39) and (54) for all $y\in\mathcal{P}_1$ and all $s>0$, respectively. □

The following corollary is the immediate consequence of Theorem 6.1 concerning the stabilities of (1).

Corollary 6.2. *Assume q and r be positive numbers. Let $g:\mathcal{P}_1\longrightarrow\mathcal{P}_2$ be an even mapping satisfying the functional inequality*

$$N\left(\mathcal{G}^1_{24}(x,y,z),s\right)\geq\begin{cases}N\left(q,s\right)\\ N\left(q\left(||x||^r+||y||^r+||z||^r\right),s\right),\\ N\left(q\ ||x||^r||y||^r||z||^r,s\right),\\ N\left(q\left(||x||^{r_1}+||y||^{r_2}+||z||^{r_3}\right),s\right),\\ N\left(q||x||^{r_1}||y||^{r_2}||z||^{r_3},s\right),\end{cases}\tag{67}$$

for all $x,y,z\in\mathcal{P}_1$ and $s>0$. Then there is a unique quadratic mapping $\mathcal{Q}_2:\mathcal{P}_1\longrightarrow\mathcal{P}_2$ and a unique quartic mapping $\mathcal{Q}_4:\mathcal{P}_1\longrightarrow\mathcal{P}_2$ such that

$$\begin{aligned}
&N\left(g_q(y)-\mathcal{Q}_2(y)-\mathcal{Q}_4(y),s\right)\\
&\geq\begin{cases}N'\left(8q,30s|4-1|\right)+N'\left(32q,30s|16-1|\right),\\
N'\left([20+4\cdot 2^r]q||y||^r,30s|4-2^r|\right)\\
+N'\left([20+4\cdot 2^r]q||y||^r,30s|16-2^r|\right), & r\neq 2,4;\\
N'\left([80+16\cdot 2^{3r}]q||y||^{3r},30s|16-2^{3r}|\right)\\
+N'\left([4+4\cdot 2^{3r}]q||y||^{3r},30s|4-2^{3r}|\right), & 3r\neq 2,4;\\
N'\left(16[1+2^{r_1}]||y||^{r_1}+32||y||^{r_2}+32||y||^{r_3},\sum_{i=1}^3 30s|16-2^{r_i}|\right)\\
+N'\left(4[1+2^{r_1}]||y||^{r_1}+8||y||^{r_2}+8||y||^{r_3},\sum_{i=1}^3 30s|4-2^{r_i}|\right) & r_1,r_2,r_3\neq 2,4;\\
N'\left(16[1+2^{r_1}]q||y|^{\sum_{i=1}^3 r_i},30s|16-2^{\sum_{i=1}^3 r_i}|\right)\\
+N'\left(4[1+2^{r_1}]q||y|^{\sum_{i=1}^3 r_i},30s|4-2^{\sum_{i=1}^3 r_i}|\right), & \sum_{i=1}^3 r_i\neq 2,4;\end{cases}
\end{aligned}\tag{68}$$

for all $y \in \mathcal{P}_1$ and $s > 0$.

7. Fuzzy Stability Results: Additive-Quadratic-Quartic Mixed Case

In this section, we establish the stability of (1).

Theorem 7.1. *Assume $a = \pm 1$ and let $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be a mapping satisfying the functional inequality*

$$N\left(\mathcal{G}^1_{24}(x,y,z), s\right) \geq N'\left(\omega\left(x,y,z\right), s\right) \tag{69}$$

for all $x, y, z \in \mathcal{P}_1$ and all $s > 0$, where $\omega, \Omega : \mathcal{P}_1^2 \longrightarrow \mathcal{P}_3$ be a mapping with the conditions (9), (10), (35), (36), (51) and (52), for all $x, y, z \in \mathcal{P}_1$ and all $s > 0$, for some $c > 0$ with $0 < \left(\frac{c}{2}\right)^a < \left(\frac{c}{4}\right)^a < \left(\frac{c}{16}\right)^a < 1$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$, a unique quadratic mapping $\mathcal{Q}_2 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ and a unique quartic mapping $\mathcal{Q}_4 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ which satisfies (1) and the inequality

$$\begin{aligned}
&N\left(g(y) - \mathcal{A}(y) - \mathcal{Q}_2(y) - \mathcal{Q}_4(y), qs\right)\\
&\quad \geq N'\left(\Omega_A\left(y\right), \frac{s|2-c|}{4}\right) + N'\left(\Omega_A\left(-y\right), \frac{s|2-c|}{4}\right)\\
&\qquad + N'\left(\Omega_Q\left(y\right), \frac{30s|4-c|}{4}\right) + N'\left(\Omega_Q\left(-y\right), \frac{30s|4-c|}{4}\right)\\
&\qquad + N'\left(\Omega_Q\left(y\right), \frac{30s|16-c|}{16}\right) + N'\left(\Omega_Q\left(-y\right), \frac{30s|16-c|}{16}\right)
\end{aligned} \tag{70}$$

where $\Omega_A\left(y\right)$, $\Omega_Q\left(y\right)$, $\mathcal{A}(y)$, $\mathcal{Q}_2(y)$ and $\mathcal{Q}_4(y)$ are defined in (12, (38) (13), (39) and (54) for all $y \in \mathcal{P}_1$ and $s > 0$, respectively.

Proof. Define a function $g_o(y)$ by $\frac{g_a(y) - g_a(-y)}{2}$, then it follows that $g_o(0) = 0$ and $g_o(-y) = -g_o(y)$ for all $y \in \mathcal{P}_1$. Thus

$$\begin{aligned}
N\left(\mathcal{G}^1_{o24}(x,y,z), s\right) &= N\left(\frac{1}{2}\left(\mathcal{G}^1_{o24}(x,y,z) - \mathcal{G}^1_{o24}(-x,-y,-z)\right), s\right)\\
&= N\left(\mathcal{G}^1_{o24}(x,y,z) - \mathcal{G}^1_{o24}(-x,-y,-z), 2s\right)\\
&\geq N\left(\mathcal{G}^1_{o24}(x,y,z), s\right) + N\left(\mathcal{G}^1_{o24}(-x,-y,-z), s\right)\\
&\geq N'\left(\omega\left(x,y,z\right), s\right) + N'\left(\omega\left(-x,-y,-z\right), s\right)
\end{aligned} \tag{71}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Using Theorem 3.1 and (71), there exists a unique additive function $\mathcal{A} : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ such that

$$N\left(g_o(y) - \mathcal{A}(y), s\right) \geq N'\left(\Omega_A\left(y\right), \frac{s|2-c|}{4}\right) + N'\left(\Omega_A\left(-y\right), \frac{s|2-c|}{4}\right) \tag{72}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. In addition, define a function $g_e(y)$ by $\dfrac{g_q(y) + g_q(-y)}{2}$, then it follows that $g_e(0) = 0$ and $g_e(-y) = g_e(y)$ for all $y \in \mathcal{P}_1$. Thus

$$\begin{aligned}
N\left(\mathcal{G}^1_{e24}(x,y,z), s\right) &= N\left(\frac{1}{2}\left(\mathcal{G}^1_{e24}(x,y,z) + \mathcal{G}^1_{e24}(-x,-y,-z)\right), s\right)\\
&= N\left(\mathcal{G}^1_{e24}(x,y,z) + \mathcal{G}^1_{e24}(-x,-y,-z), 2s\right)\\
&\geq N\left(\mathcal{G}^1_{e24}(x,y,z), s\right) + N\left(\mathcal{G}^1_{e24}(-x,-y,-z), s\right)\\
&\geq N'\left(\omega\left(x,y,z\right), s\right) + N'\left(\omega\left(-x,-y,-z\right), s\right) \qquad (73)
\end{aligned}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Using Theorem 6.1 and (73), there exists a unique quadratic mapping $\mathcal{Q}_2 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ and a unique quartic mapping $\mathcal{Q}_4 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ such that

$$\begin{aligned}
N\left(g_e(y) - \mathcal{Q}_2(y) - \mathcal{Q}_4(y), s\right) \geq{}& N'\left(\Omega_Q\left(y\right), \frac{30s|4-c|}{4}\right) + N'\left(\Omega_Q\left(-y\right), \frac{30s|4-c|}{4}\right)\\
&+ N'\left(\Omega_Q\left(y\right), \frac{30s|16-c|}{16}\right) + N'\left(\Omega_Q\left(-y\right), \frac{30s|16-c|}{16}\right) \qquad (74)
\end{aligned}$$

for all $y \in \mathcal{P}_1$ and $s > 0$. Define

$$g(y) = g_o(y) + g_e(y) \qquad (75)$$

for all $y \in \mathcal{P}_1$. Now from (75), (74) and (72), we arrive our result. $\square$

Corollary 7.2. *Assume q and r be positive numbers. Let $g : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be a mapping satisfying the functional inequality*

$$N\left(\mathcal{G}^1_{24}(x,y,z), s\right) \geq \begin{cases} N\left(q, s\right)\\ N\left(q\left(||x||^r + ||y||^r + ||z||^r\right), s\right),\\ N\left(q\ ||x||^r||y||^r||z||^r, s\right),\\ N\left(q\left(||x||^{r_1} + ||y||^{r_2} + ||z||^{r_3}\right), s\right),\\ N\left(q||x||^{r_1}||y||^{r_2}||z||^{r_3}, s\right), \end{cases} \qquad (76)$$

for all $x, y, z \in \mathcal{P}_1$ and $s > 0$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$, a unique quadratic mapping $\mathcal{Q}_2 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ and a unique quartic *mapping* $\mathcal{Q}_4 : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ *such that*

$$N\left(g_q(y) - \mathcal{A}(y) - \mathcal{Q}_2(y) - \mathcal{Q}_4(y), s\right)$$

$$\geq \begin{cases} N'\left(8q, s|2-1|\right) + N'\left(8q, 30s|4-1|\right) + N'\left(32q, 30s|16-1|\right), & \\ N'\left(24q||y||^r, s|2-2^r|\right) & \\ +N'\left([20+4\cdot 2^r]q||y||^r, 30s|4-2^r|\right) & \\ +N'\left([20+4\cdot 2^r]q||y||^r, 30s|16-2^r|\right), & r \neq 1,2,4; \\ N'\left(8q||y||^{3r}, s|2-2^{3r}|\right) & \\ +N'\left([80+16\cdot 2^{3r}]q||y||^{3r}, 30s|16-2^{3r}|\right) & \\ +N'\left([4+4\cdot 2^{3r}]q||y||^{3r}, 30s|4-2^{3r}|\right), & 3r \neq 1,2,4; \\ N'\left(8q\sum_{i=1}^{3}||y||^{r_i}, \sum_{i=1}^{3} s|2-2^{r_i}|\right) & \\ N'\left(16[1+2^{r_1}]||y||^{r_1} + 32||y||^{r_2} + 32||y||^{r_3}, \sum_{i=1}^{3} 30s|16-2^{r_i}|\right) & \\ +N'\left(4[1+2^{r_1}]||y||^{r_1} + 8||y||^{r_2} + 8||y||^{r_3}, \sum_{i=1}^{3} 30s|4-2^{r_i}|\right) & r_1, r_2, r_3 \neq 1,2,4; \\ N'\left(8q||y|^{\sum_{i=1}^{3} r_i}, s|2-2^{\sum_{i=1}^{3} r_i}|\right) & \\ +N'\left(16[1+2^{r_1}]q||y|^{\sum_{i=1}^{3} r_i}, 30s|16-2^{\sum_{i=1}^{3} r_i}|\right) & \\ +N'\left(4[1+2^{r_1}]q||y|^{\sum_{i=1}^{3} r_i}, 30s|4-2^{\sum_{i=1}^{3} r_i}|\right), & \sum_{i=1}^{3} r_i \neq 2,4; \end{cases} \tag{77}$$

for all $y \in \mathcal{P}_1$ *and all* $s > 0$.

Acknowledgment

The authors sincerely thank the anonymous reviewer for her/his careful reading, constructive comments to improve the quality of the first draft of paper.

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