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Partially ordered cone metric spaces and coupled fixed point theorems via α-series

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Abstract. This research tends to focus on proving the results of coupled fixed point in partially ordered cone metric spaces by imposing some condition on a self-mapping and a sequence of mappings via α-series. The α-series are wider than the convergent series. Furthermore, an example is provided to illustrate the results.

1. Introduction

The concept of cone metric space was introduced in [4] by replacing an ordered Banach space instead of real numbers and proved some fixed point theorems for contractive mappings over cone metric spaces. Later, in different methods, their fixed point theorems were generalized by many authors. In [6, 8] coincidence point theory over cone metric spaces are studied. The concept of a coupled coincidence point was introduced in [9] and they studied fixed point theorems in partially ordered metric spaces. Shatanawi [13] proved that coupled coincidence point theorems over cone metric spaces are not necessarily normal.

In this article, we establish the results of coupled fixed point for a self mapping g and $\{T_i\}_{i\in\mathbb{N}}$ that is a sequence of mappings from $X^2 \to X$, where $\mathbb{N}$ is positive integer, in partially ordered cone metric spaces via α-series, that introduced in [12]. The α-series are wider than the convergent series. This research is a generalization and combination of some articles, which are as references for it. We provide the preliminaries and definitions used throughout the article.

Definition 1.1. ([4]) Let $P \subseteq E$, where E is a real Banach space with $int(P) \neq \emptyset$. Then, P is called a cone if the following conditions are satisfied:

1. P is closed and $P \neq \{\theta\}$, where θ represents zero.
2. $a, b \in \mathbb{R}^+, x^1, x^2 \in P$ implies $ax + by \in P$.

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3. $x^1 \in P \cap -P$ implies $x^1 = \theta$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to P by $x^1 \leq x^2$ iff $x^2 - x^1 \in P$. We write $x^1 < x^2$ to show that $x^1 \leq x^2$ but $x^1 \neq x^2$. We write $x^1 \ll x^2$ if $x^2 - x^1 \in IntP$. It is easy to show that $\lambda Int(P) \subseteq Int(P)$ for all positive scalar λ.

Definition 1.2. ([4]) Let $X \neq \emptyset$. Suppose the mapping $d : X^2 \to E$ satisfies
1. $\theta \leq d(x^1, x^2)$ for all $x^1, x^2 \in X$ and $d(x^1, x^2) = \theta$ iff $x^1 = x^2$.
2. $d(x^1, x^2) = d(x^2, x^1)$ for all $x^1, x^2 \in X$.
3. $d(x^1, x^2) \leq d(x^1, x^3) + d(x^3, x^2)$ for all $x^1, x^2, x^3 \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.3. ([4]) Let (X, d) be a cone metric space, $\{x^1_n\}$ be a sequence in X and $x^1 \in X$.
(i) For every $c \in E$ with $\theta \ll c$, there is an $N \in \mathbb{N}$ such that $d(x^1_n, x^1) \ll c$ for all $n \geq \mathbb{N}$, then$\{x^1_n\}$ is called *converges* to x^1. We denote this by $lim_{n\to+\infty} x^1_n = x^1$.
(ii) For every $c \in E$ with $\theta \ll c$, there is an $N \in \mathbb{N}$ such that $d(x^1_n, x^1_m) \ll c$ for all $n, m \geq \mathbb{N}$, then $\{x^1_n\}$ is called a *Cauchy sequence* in X.
(iii) The space (X, d) is called a *complete cone metric space* if every Cauchy sequence is convergent.

Definition 1.4. ([2]) Let (X, d) be a cone metric space and $x^1_0 \in X$. Then, a mapping $f : X \to X$ is said to be *continuous at* x^1_0 if for any sequence $x^1_n \to x^1_0$, we have $fx^1_n \to fx^1_0$.

Definition 1.5. ([1]) An element $(x^1, x^2) \in X^2$ is called a *coupled fixed point* of $F : X^2 \to X$ if

$$F(x^1, x^2) = x^1, \quad F(x^2, x^1) = x^2.$$

Example 1.6. Consider $X = [0, \infty)$ and $F : X^2 \to X$ is defined by $F(x^1, x^2) = x^1x^2$ for all $x^1, x^2 \in X$. One can easily see that F has a unique coupled fixed point $(1, 1)$.

Definition 1.7. ([11]) An element $(x^1, x^2) \in X^2$ is called a *coupled coincidence point* of the mappings $F : X^2 \to X$ and $g : X \to X$ if $F(x^1, x^2) = x^1$ and $F(x^2, x^1) = gx^2$. In this case, (x^1, gx^2) is called a *coupled coincidence point.*

Definition 1.8. ([5]) An element $(x^1, x^2) \in X^2$ is called a *common coupled fixed point* of mappings $F : X^2 \to X$ and $g : X \to X$ if $x^1 = g(x^1) = F(x^1, x^2)$ and $x^2 = g(x^2) = F(x^2, x^1)$.

Definition 1.9. ([3]) Let $X \neq \emptyset$. We say the mappings $F : X^2 \to X$ and $g : X \to X$ are *commutative* if

$$gF(x^1, x^2) = F(gx^1, gx^2), \quad gF(x^2, x^1) = F(gx^2, gx^1)$$

Now, inspire [14], we generalize the concept of compatible mapping for a self-mapping g and a bivariate mapping F on a cone metric space as follows.

Definition 1.10. The mappings $F : X^2 \to X$ and $g : X \to X$ are called *compatible* if for arbitrary $c \in intP$, there exists $n_0 \in \mathbb{N}$ such that

$$d(gF(x_n^1, x_n^2), F(gx_n^1, gx_n^2)) \ll c,$$
$$d(gF(x_n^2, x_n^1), F(gx_n^2, gx_n^1)) \ll c$$

whenever $n > n_0$; $\{x_n^1\}, \{x_n^2\} \in X$, such that

$$\lim_{n\to+\infty} F(x_n^1, x_n^2) = \lim_{n\to+\infty} gx_n^1 = x^1,$$
$$\lim_{n\to+\infty} F(x_n^2, x_n^1) = \lim_{n\to+\infty} gx_n^2 = x^2,$$

for some $x^1, x^2 \in X$. It is also said to be *weakly compatible* if they commute at coincidence points.

Example 1.11. Let $X = [0, 3]$ be endowed with $d(x^1, x^2) = |x^1 - x^2|$. Define $F : X^2 \to X$ and $g : X \to X$ by

$$F(x^1, x^2) = \begin{cases} x^1 + x^2 & \text{if } x^1, x^2 \in [0, 1) \\ 3 & \text{elsewhere.} \end{cases}$$

$$g(x^1) = \begin{cases} x^1 & \text{if } x^1 \in [0, 1) \\ 3 & \text{if } x^1 \in [1, 3] \end{cases}$$

Then for any $x^1, x^2 \in [1, 3]$, $F(gx^1, gx^2) = gF(x^1, x^2)$ and $F(gx^2, gx^1) = gF(x^2, x^1)$, show that F, g are weakly compatible maps on $[0, 3]$.

Example 1.12. Let $X = \mathbb{R}$ be endowed with the usual metric space $d(x^1, x^2) = |x^1 - x^2|$. Define $F : X^2 \to X$ and $g : X \to X$ by

$$F(x^1, x^2) = x^1 + x^2, \quad g(x^1) = (x^1)^2$$

Then F and g are not weakly compatible maps on $\mathbb{R}$.

Definition 1.13. ([14]) The mappings $F : X^2 \to X$ and $g : X \to X$ are called *reciprocally continuous* if

$$\lim_{n\to+\infty} gF(x_n^1, x_n^2) = gx^1, \ \text{and} \ \lim_{n\to+\infty} F(gx_n^1, gx_n^2) = F(x^1, x^2)$$
$$\lim_{n\to+\infty} gF(x_n^2, x_n^1) = gx^2, \ \text{and} \ \lim_{n\to+\infty} F(gx_n^2, gx_n^1) = F(x^2, x^1)$$

whenever $\{x_n^1\}, \{x_n^2\} \in X$, such that

$$\lim_{n\to+\infty} F(x_n^1, x_n^2) = \lim_{n\to+\infty} gx_n^1 = x^1,$$

$$\lim_{n\to+\infty} F(x_n^2, x_n^1) = \lim_{n\to+\infty} gx_n^2 = x^2,$$

for some $x^1, x^2 \in X$.

Definition 1.14. ([10]) Let $(X, \preceq)$ be a partially ordered set and $F : X^2 \to X$. We say that F has the *mixed monotone property* if for any $x^1, x^2 \in X$

$$\begin{array}{lll} x_1^1, x_2^1 \in X, \quad x_1^1 \preceq x_2^1 & \Rightarrow & F(x_1^1, x^2) \preceq F(x_2^1, x^2), \\ x_1^2, x_2^2 \in X, \quad x_1^2 \preceq x_2^2 & \Rightarrow & F(x^1, x_1^2) \succeq F(x^1, x_2^2), \end{array}$$

That is, $F(x^1, x^2)$ is monotone non-decreasing in x^1 and is monotone non-increasing in x^2.

The concept of the mixed monotone property is generalized in [11] as follows.

Definition 1.15. ([7]) Let $(X, \preceq)$ be a partially ordered set and $F : X^2 \to X$ and $g : X \to X$. We say that F has the *g-mixed monotone property* if for any $x^1, x^2 \in X$,

$$\begin{array}{lll} x_1^1, x_2^1 \in X, \quad gx_1^1 \preceq gx_2^1 & \Rightarrow & F(x_1^1, x^2) \preceq F(x_2^1, x^2), \\ x_1^2, x_2^2 \in X, \quad gx_1^2 \preceq gx_2^2 & \Rightarrow & F(x^1, x_1^2) \succeq F(x^1, x_2^2), \end{array}$$

That is, $F(x^1, x^2)$ is monotone non-decreasing in x^1, and it is monotone non-increasing in x^2.

The new concept of an α-series is introduced by Sihag et al. [12] as follow.

Definition 1.16. ([12]) Let $\{a_n\}$ be a sequence of non-negative real numbers. We say that a series $\sum_{n=1}^{+\infty} a_n$ is an α-series, if there exist $0 < \alpha < 1$ and $n_\alpha \in \mathbb{N}$ such that $\sum_{i=1}^{k} a_i \leq \alpha k$ for each $k \geq n_\alpha$.

For example, we know that every convergent series is bounded hence every convergent series of non-negative real terms is an α-series. Moreover, there exists also divergent series that are α-series. For example, $\sum_{n=1}^{+\infty} \dfrac{1}{n}$ is an α-series.

2. Main results

In this section, we will wish to study existence and uniqueness of coupled common fixed point for sequence of mappings $T_n : X^2 \to X$ and $g : X \to X$, where (X, d) is a cone metric space. In view of Definition 1.15, we have the next definition.

Definition 2.1. Let $(X, \preceq)$ be a partially ordered set and $T_n : X^2 \to X,\ n \in \mathbb{N}$ and $g : X \to X$ are given. We say that $\{T_n\}$ has *g-mixed monotone property* if for any $x^1, x'^1, x^2, x'^2 \in X$,

$$\begin{aligned} gx^1 \preceq gx'^1,\ gx'^2 \preceq gx^2 \ \textit{imply}\ & T_n(x^1, x^2) \preceq T_{n+1}(x'^1, x'^2), \\ & T_n(x^2, x^1) \succeq T_{n+1}(x'^2, x'^1). \end{aligned} \tag{2.1}$$

Definition 2.2. Let $T_i : X^2 \to X$ and $g : X \longrightarrow X$. We call $\{T_i\}_{i\in\mathbb{N}\cup\{0\}}$ and g are satisfied in (K) property if there exists $0 \leq \beta_{i,j}, \gamma_{i,j} < 1$ for $i, j \in \mathbb{N}$, such that

$$\begin{aligned} d(T_i(x^1, x^2), T_j(u^1, u^2)) \leq \beta_{i,j}[d(gx^1, T_i(x^1, x^2)) + d(gu^1, T_j(u^1, u^2))] \\ + \gamma_{i,j} d(gu^1, gx^1) \end{aligned} \tag{2.2}$$

for $x^1, x^2, u^1, u^2 \in X$ with $x^1 \succeq u^1, u^2 \succeq x^2$ *or* $gx^1 \succeq gu^1, gu^2 \succeq gx^2$, $0 \leq \beta_{i,j}, \gamma_{i,j} < 1$ *for* $i, j \in \mathbb{N}$. which $\sum_{i=1}^{+\infty} \left(\dfrac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}}\right)$ be an α-series.

Definition 2.3. We call T_0 and g have a non-decreasing transcendence point in the first component and non-increasing transcendence point in the second component, which we call T_0 and g have *mixed coupled transcendence point*, if there exist $x_0^1, x_0^2 \in X$ such that

$$T_0(x_0^1, x_0^2) \succeq gx_0^1,\ T_0(x_0^2, x_0^1) \preceq gx_0^2. \tag{2.3}$$

Before presenting the main result, we first consider the sequences that are made in the upcoming lemma.

Lemma 2.1. *Let $(X, d, \preceq)$ be a partially ordered cone metric space and g and $\{T_i\}_{i\in\mathbb{N}}$ are given. $\{T_i\}_{i\in\mathbb{N}}$ has a g-mixed monotone property with $T_i(X^2) \subseteq g(X)$. If T_0 and g have mixed coupled transcendence point, then*

(a) there are sequences $\{x_n^1\}$ and $\{x_n^2\}$ in X such that

$$gx_n^1 = T_{n-1}(x_{n-1}^1, x_{n-1}^2),\ gx_n^2 = T_{n-1}(x_{n-1}^2, x_{n-1}^1).$$

(b) $\{gx_n^1\}$ is a non-decreasing and $\{gx_n^2\}$ is a non-increasing sequences.

(c) If $\{T_i\}_{i\in\mathbb{N}}$ and g satisfy the condition (K), then $\{gx_n^1\}$ and $\{gx_n^2\}$ are Cauchy sequences.

Proof. By hypothesis, let $x_0^1, x_0^2 \in X$ such that condition 2.3 holds. Since $T_i(X^2) \subseteq g(X)$, we can define $x_1^1, x_1^2 \in X$ such that $gx_1^1 = T_0(x_0^1, x_0^2)$ and $gx_1^2 = T_0(x_0^2, x_0^1)$. Again since $T_i(X^2) \subseteq g(X)$, there exists $x_2^1, x_2^2 \in X$ such that $gx_2^1 = T_1(x_1^1, x_1^2)$ and $gx_2^2 = T_1(x_1^2, x_1^1)$. We have $gx_0^1 \preceq gx_1^1 \preceq gx_2^1$ and $gx_2^2 \preceq gx_1^2 \preceq gx_0^2$, since $\{T_i\}_{i\in\mathbb{N}}$ has the mixed g-monotone property. Continuing this technique, we can construct sequences $\{x_n^1\}, \{x_n^2\} \in X$ such that

$$gx_n^1 = T_{n-1}(x_{n-1}^1, x_{n-1}^2) \preceq gx_{n+1}^1 = T_n(x_n^1, x_n^2)$$

and

$$gx_{n+1}^2 = T_n(x_n^2, x_n^1) \preceq gx_n^2 = T_{n-1}(x_{n-1}^2, x_{n-1}^1)$$

for all $n \in \mathbb{N}$, which prove (a) and (b). To prove (c), we consider the sequences $\{x_n^1\}$ and $\{x_n^2\}$ constructed above. Then by condition (2.2), we have

$$\begin{aligned} d(gx_n^1, gx_{n+1}^1) &= d(T_{n-1}(x_{n-1}^1, x_{n-1}^2), T_n(x_n^1, x_n^2)) \\ &\leq \beta_{n-1,n}[d(gx_{n-1}^1, T_{n-1}(x_{n-1}^1, x_{n-1}^2)) + d(gx_n^1, T_n(x_n^1, x_n^2))] \\ &+ \gamma_{n-1,n} d(gx_n^1, gx_{n-1}^1) \\ &= \beta_{n-1,n}[d(gx_{n-1}^1, gx_n^1) + d(gx_n^1, gx_{n+1}^1)] \\ &+ \gamma_{n-1,n} d(gx_n^1, gx_{n-1}^1). \end{aligned}$$

It follows that

$$(1-\beta_{n-1,n}) d(gx_n^1, gx_{n+1}^1) \leq (\beta_{n-1,n} + \gamma_{n-1,n}) d(gx_{n-1}^1, gx_n^1).$$

We get

$$\begin{aligned} d(gx_n^1, gx_{n+1}^1) &\leq \left(\frac{\beta_{n-1,n}+\gamma_{n-1,n}}{1-\beta_{n-1,n}}\right) d(gx_{n-1}^1, gx_n^1) \\ &\leq \left(\frac{\beta_{n-1,n}+\gamma_{n-1,n}}{1-\beta_{n-1,n}}\right)\left(\frac{\beta_{n-2,n-1}+\gamma_{n-2,n-1}}{1-\beta_{n-2,n-1}}\right) d(gx_{n-2}^1, gx_{n-1}^1) \\ &\vdots \\ &\leq \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) d(gx_0^1, gx_1^1) \end{aligned} \tag{2.4}$$

The same as above, we can also show that

$$d(gx_n^2, gx_{n+1}^2) \leq \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) d(gx_0^2, gx_1^2) \tag{2.5}$$

Plugging (2.4) into (2.5), we have

$$\begin{aligned} \delta_n &:= d(gx_n^1, gx_{n+1}^1) + d(gx_n^2, gx_{n+1}^2) \\ &\leq \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)[d(gx_0^1, gx_1^1) + d(gx_0^2, gx_1^2)] \\ &= \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)\delta_0 \end{aligned}$$

Let $m, n \in \mathbb{N}$ with $m > n$, and α and n_α as in Definition 1.16, then for $n \geq n_\alpha$ also, with repeated use of the triangle inequality, we obtain

$$\begin{aligned}
d(gx_n^1, gx_m^1) + d(gx_n^2, gx_m^2) &\leq d(gx_n^1, gx_{n+1}^1) + d(gx_n^2, gx_{n+1}^2) \\
&+ d(gx_{n+1}^1, gx_{n+2}^1) + d(gx_{n+1}^2, gx_{n+2}^2) \\
&+ \ldots + d(gx_{m-1}^1, gx_m^1) + d(gx_{m-1}^2, gx_m^2) \\
&\leq \prod_{i=0}^{n-1} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0 + \prod_{i=0}^{n} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0 \\
&+ \ldots + \prod_{i=0}^{m-2} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0 \\
&= \sum_{k=0}^{m-n-1} \prod_{i=0}^{n+k-1} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0 \\
&= \sum_{k=n}^{m-1} \prod_{i=0}^{k-1} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0 \\
&\leq \sum_{k=n}^{m-1} \left[\frac{1}{k} \sum_{i=0}^{k-1} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \right]^k \delta_0 \\
&\leq \left(\sum_{k=n}^{m-1} \alpha^k \right) \delta_0 \\
&\leq \frac{\alpha^n}{1 - \alpha} \delta_0..
\end{aligned}$$

Now, we prove that $\{gx_n^1\}, \{gx_n^2\}$ are Cauchy sequences in (X, d). Let $\theta \ll c$ be given. There is a neighborhood of θ

$$N_\delta(\theta) = \{r \in E : \parallel r \parallel < \delta\}$$

where $\delta > 0$, such that $c + N_\delta(\theta) \subseteq IntP$, since $c \in IntP$. Choose $N_1 \in \mathbb{N}$ such that

$$\parallel -\frac{\alpha^{N_1}}{1 - \alpha} \delta_0 \parallel < \delta.$$

Then

$$-\frac{\alpha^n}{1 - \alpha} \delta_0 \in N_\delta(\theta)$$

for all $n \geq N_1$. Hence

$$c - \frac{\alpha^n}{1 - \alpha} \delta_0 \in c + N_\delta(\theta) \subseteq IntP.$$

Thus we have

$$\frac{\alpha^n}{1 - \alpha} \delta_0 \ll c$$

for all $n \geq N_1$. Therefore

$$d(gx_n^1, gx_m^1) + d(gx_n^2, gx_m^2) \leq \frac{\alpha^n}{1-\alpha}\delta_0 \ll c$$

for all $m > n \geq N_1$. So we conclude $\{gx_n^1\}$ and $\{gx_n^2\}$ are Cauchy in $g(X)$. □

Now, we revise Definitions 1.10 and 1.13.

Definition 2.4. Let (X, d) be a cone metric space. The mappings $\{T_i\}_{i\in\mathbb{N}}$ and g are *compatible*, if for arbitrary $c \in intP$, there exists $n_0 \in \mathbb{N}$ such that

$$d(g(T_n(x_n^1, x_n^2), T_n(gx_n^1, gx_n^2)) \ll c,$$
$$d(g(T_n(x_n^2, x_n^1), T_n(gx_n^2, gx_n^1)) \ll c$$

whenever $n > n_0$, $\{x_n^1\}, \{x_n^2\} \in X$, such that

$$\lim_{n\to+\infty} T_n(x_n^1, x_n^2) = \lim_{n\to+\infty} gx_{n+1}^1 = x^1,$$
$$\lim_{n\to+\infty} T_n(x_n^2, x_n^1) = \lim_{n\to+\infty} gx_{n+1}^2 = x^2,$$

for some $x^1, x^2 \in X$, It is said to be *weakly compatible* if $gx^1 = T_i(x^1, x^2)$ and $gx^2 = T_i(x^2, x^1)$ implies

$$g(T_i(x^1, x^2)) = T_i(gx^1, gx^2),\ g(T_i(x^2, x^1)) = T_i(gx^2, gx^1),\ for\ \ i \in \mathbb{N}$$

.

Definition 2.5. The mappings $T_n : X^2 \to X$ and $g : X \to X$ are called *reciprocally continuous* if

$$\lim_{n\to+\infty} gT_n(x_n^1, x_n^2) = gx^1, and \lim_{n\to+\infty} T_n(gx_n^1, gx_n^2) = \lim_{n\to+\infty} T_n(x^1, x^2)$$
$$\lim_{n\to+\infty} gT_n(x_n^2, x_n^1) = gx^2, and \lim_{n\to+\infty} T_n(gx_n^2, gx_n^1) = \lim_{n\to+\infty} T_n(x^2, x^1)$$

whenever $\{x_n^1\}, \{x_n^2\} \in X$, such that

$$\lim_{n\to+\infty} T_n(x_n^1, x_n^2) = \lim_{n\to+\infty} gx_{n+1}^1 = x^1,$$
$$\lim_{n\to+\infty} T_n(x_n^2, x_n^1) = \lim_{n\to+\infty} gx_{n+1}^2 = x^2,$$

for some $x^1, x^2 \in X$.

Theorem 2.2. *In addition to the assumptions of Lemma 2.1, let $\{T_i\}_{i\in\mathbb{N}}$ and g be reciprocally continuous, compatible, g is continuous and $g(X) \subseteq X$ is complete. Also, suppose that X has the following properties:*

1. if a increasing sequence $x_n^1 \to x^1$, then $x_n^1 \preceq x^1$ for all n,

2. if a decreasing sequence $x_n^1 \to x^1$, then $x^1 \preceq x_n^1$ for all n,

then $\{T_i\}_{i\in\mathbb{N}}$ and g have a coupled coincidence point.

Proof. Let $\{x_n^1\}$ and $\{x_n^2\}$ are the some sequence which appear in the Lemma 2.1. Since $g(X)$ is complete, then there exist $(x^1, x^2) \in X^2$ such that

$$\lim_{r\to+\infty}\{x_n^1\} = gx^1 := x^1, \quad \lim_{r\to+\infty}\{gx_n^2\} = gx^2 := x^2.$$

We now have

$$\lim_{n\to+\infty} T_n(x_n^1, x_n^2) = \lim_{n\to+\infty} gx_{n+1}^1 = x^1,$$
$$\lim_{n\to+\infty} T_n(x_n^2, x_n^1) = \lim_{n\to+\infty} gx_{n+1}^2 = x^2.$$

Since $\{T_i\}_{i\in\mathbb{N}}$ and g are weakly reciprocally continuous, we have

$$\lim_{n\to+\infty} gT_n(x_n^1, x_n^2) = gx^1, and \lim_{n\to+\infty} T_n(gx_n^1, gx_n^2) = \lim_{n\to+\infty} T_n(x^1, x^2)$$
$$\lim_{n\to+\infty} gT_n(x_n^2, x_n^1) = gx^2, and \lim_{n\to+\infty} T_n(gx_n^2, gx_n^1) = \lim_{n\to+\infty} T_n(x^2, x^1)$$

besides, by the compatibility of $\{T_i\}_{i\in\mathbb{N}}$ and g we have

$$\lim_{n\to\infty} T_n(gx_n^1, gx_n^2) = gx^1,$$
$$\lim_{n\to\infty} T_n(gx_n^2, gx_n^1) = gx^2.$$

Also from continuity of g, we have

$$\lim_{n\to\infty} T_n(x^1, x^2) = gx^1,$$
$$\lim_{n\to\infty} T_n(x^2, x^1) = gx^2$$

from the non-decreasing sequence $\{gx_n^1\}$ and $gx_n^1 \to x^1$, we have $gx_n^1 \preceq x^1$ for all n. Also, from the non-decreasing sequence $\{gx_n^2\}$ and $gx_n^2 \to x^2$, we have $x^2 \preceq gx_n^2$ for all n. Then applying condition (2.2), we get

$$d(T_i(x^1, x^2), T_n(gx_n^1, gx_n^2) \leq \beta_{i,n}[d(gx^1, T_i(x^1, x^2)) + d(gx_n^1, T_n(gx_n^1, gx_n^2))] + \gamma_{i,n} d(gx_n^1, gx^1)$$

Let $\theta \ll c$ be given. Choose $N_1, N_2 \in \mathbb{N}$ such that

$$\beta_{i,n}[d(gx^1, T_i(x^1, x^2)) + d(gx_n^1, T_n(gx_n^1, gx_n^2))] \ll \frac{c}{2} \quad \textit{for all } n \geq N_1,$$

as $\beta_{i,n} < 1$, and

$$\gamma_{i,n} d(gx_n^1, gx^1)) \ll \frac{c}{2} \quad \textit{for all } n \geq N_2.$$

Let $N_0 = \max\{N_1, N_2\}$. Then

$$\beta_{i,n}[d(gx^1, T_i(x^1, x^2)) + d(gx_n^1, T_n(gx_n^1, gx_n^2)] + \gamma_{i,n} d(gx_n^1, gx^1) \ll c$$

for all $n \geq N_0$. Hence $T_n(gx_n^1, gx_n^2)$ converges to $T_i(x^1, x^2)$. Let $\theta \ll c$ be given. We choose $k_1, k_2, k_3 \in \mathbb{N}$ such that

$$d(T_n(x^1, x^2), T_n(gx_n^1, gx_n^2))) \ll \frac{c}{3} \textit{ for all } n \geq k_1,$$

$$d(T_n(gx_n^1, gx_n^2), gT_n(x_n^1, x_n^2)) \ll \frac{c}{3} \textit{ for all } n \geq k_2$$

and

$$d(gT_n(x_n^1, x_n^2), x^1) \ll \frac{c}{3} \textit{ for all } n \geq k_3$$

Let $k_0 = \max\{k_1, k_2, k_2\}$. Then $d(T_i(x^1, x^2), x^1) \ll c$. Since c is arbitrary, we have

$$d(T_i(x^1, x^2), x^1) \ll \frac{c}{m} \quad \forall m \in \mathbb{N}.$$

Letting $m \to +\infty$, we find $\frac{c}{m} \to \theta$ and so we conclude that

$$\frac{c}{m} - d(T_i(x^1, x^2), x^1)) \to -d(T_i(x^1, x^2), x^1), \textit{ as } m \to +\infty.$$

Since P is closed, we get $-d(T_i(x^1, x^2), x^1) \in P$. Thus $d(T_i(x^1, x^2), x^1) \in P \cap -P$. Hence $d(T_i(x^1, x^2), x^1) = \theta$. Therefore $gx^1 = T_i(x^1, x^2)$. In the same way, we may prove that $gx^2 = T_i(x^2, x^1)$. Thus, (x^1, x^2) is a coupled coincidence point of $\{T_i\}_{i \in \mathbb{N}}$ and g. □

Definition 2.6. For $x^1, x^2 \in X$, we say that (x^1, x^2) is coupled comparable with (u^1, u^2) if and only if

$$x^1 \succeq u^1, x^2 \preceq u^2 \textit{ or } x^1 \preceq u^1, x^2 \succeq u^2 \textit{ or}$$
$$x^1 \succeq u^2, x^2 \preceq u^1 \textit{ or } x^1 \preceq u^2, x^2 \succeq u^1.$$

If in the above definition replace (x^1, x^2) and (u^1, u^2) with (gx^1, gx^2) and (gu^1, gu^2), we call (x^1, x^2) is coupled comparable with (u^1, u^2) with respect to g.

Theorem 2.3. *Let $(X, d, \preceq)$ be a partially ordered con metric space. Let g and $\{T_i\}_{i \in \mathbb{N}}$ are given. $\{T_i\}_{i \in \mathbb{N}}$ and g are w-compatible and satisfy the condition (K). If $\{T_i\}_{i \in \mathbb{N}}$ have coupled coincidence points comparable with respect to g, then $\{T_i\}_{i \in \mathbb{N}}$ and g have a unique coupled common fixed point, that is, there exists unique $(x^1, x^2) \in X^2$ such that $x^1 = gx^1 = T_i(x^1, x^2), x^2 = gx^2 = T_i(x^2, x^1)$ for $i \in \mathbb{N}$. Moreover, common fixed point of $\{T_i\}_{i \in \mathbb{N}}$ and g is of the form (p, p) for some $p \in X$.*

Proof. From Theorem 2.2, the set of coupled coincidence points is non-empty. First, we show that if (x^1, x^2) and (u^1, u^2) are coupled coincidence points, then $gx^1 =$

$gu^1, gx^2 = gu^2$. Since the set of coupled coincidence points is coupled comparable, applying condition (2.2), we get

$$\begin{aligned} d(gx^1, gu^1) &= d(T_i(x^1, x^2), T_j(u^1, u^2)) \\ &\leq \beta_{i,j}[d(gx^1, T_i(x^1, x^2)) + d(gu^1, T_j(u^1, u^2))] + \gamma_{i,j}(d(gu^1, gx^1)), \end{aligned}$$

(If necessary, commute x^1 and gu) and so as $\gamma_{i,j} < 1$, it follows that $d(gx^1, gu^1) = 0$, that is, $gx^1 = gu^1$. In the same way, we can prove $gx^2 = gu^2$, and also $gx^1 = gu^2$ and $gx^2 = gu^1$. Thus $gx^1 = gx^2$. Therefore $\{T_i\}_{i\in\mathbb{N}}$ and g have unique coupled point of coincidence (gx^1, gx^1). Now, let $gx^1 = p$. Then we have $p = gx^1 = T_i(x^1, x^1)$. By w-compatibility of $\{T_i\}_{i\in\mathbb{N}}$ and g, we have

$$gp = ggx^1 = g(T_i(x^1, x^1)) = T_i(gx^1, gx^1) = T_i(p, p) = gx^1$$

Then (gp, gp) is coupled point of coincidence of $\{T_i\}_{i\in\mathbb{N}}$ and g, and $gp = gx^1$. Therefore $p = gp = T_i(p, p)$. Hence (p, p) is unique coupled common fixed point of $\{T_i\}_{i\in\mathbb{N}}$ and g.

□

Example 2.7. Let $X = [0, 1]$, and

$$P = \{(x^1, x^2) \in \mathbb{R}^2 : x^1, x^2 \geq 0\} \subseteq E = \mathbb{R}^2.$$

Define $d(x^1, x^2) = (|x^1 - x^2|, |x^1 - x^2|)$. Then (X, d) is a partially ordered complete cone metric space.
Define $\beta_{i,j} = \frac{1}{2^{2i+1}}, \gamma_{i,j} = \frac{1}{2^i}$ for all $i, j = 1, 2, \ldots$ and consider the mapping $T_i : X^2 \to X$ and $g : X \to X$ with

$$T_i(x^1, x^2) = \frac{x^1 + x^2}{2^i}, \; g(x^1) = 12x^1$$

for all $x^1, x^2 \in X$, $i = 1, 2, \ldots$.
For $x^1 \geq u^1, x^2 \leq u^2$ and for all $i, j \in \mathbb{N}$, it can be easily verified by mathematical induction that the inequality (2.2) holds for all $x^1, x^2, u^1, u^2 \in X$. Moreover, the series

$$\sum_{i=1}^{+\infty} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) = \sum_{i=1}^{+\infty} \frac{2^{i+1} + 1}{2^{2i+1} - 1}$$

is an α-series with $\alpha = \frac{1}{2}$. So all conditions of Theorem 2.2 are satisfied and $(0, 0)$ are the coupled coincident points of g and T_i. Moreover, using the same g and T_i in Theorem 2.3, (0, 0) is the unique coupled common fixed point of g and T_i.

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